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Design of elastic transformation media via affine transformation in curvilinear coordinates

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ABSTRACT

In this paper, the transformation medium theory in elastodynamics is studied. We discuss its availability and propose an affine transformation to approach the transformed fields together with material parameters. Explicit constitutive relations of materials and physical fields between original and transformed space are presented in curvilinear coordinates. Comparing to the literature, the transformed mass density derived here is a scalar rather than a diagonal tensor and the corresponding elasticity tensor possesses full symmetries, which means that they are more practicable for application.

1. INTRODUCTION

In recent years, the use of coordinate transformations to design material specifications that control the propagation of electromagnetic waves as desired has been discussed. The basic idea can be referred to two pioneered works from Leonhardt (2006) and Pendry (2006). The key idea to achieve this effect is that, with appropriately designed material parameters, the governing equations will remain unchanged in form under coordinate transformation. The mathematical technique for manipulating electromagnetic waves is called transformation optics whereas the materials are named transformation medium. In addition to optics, similar concepts could be extended to other physical phenomena such as acoustics (Chen 2007, Cummer 2007 & 2008, Norris 2008), conductivity (Chen 2007) and quantum mechanics [Zhang 2008, Greenleaf 2008, Lin 2009]. However, related researches on transformation medium reported so far pay less attention to the application of elastodynamics. Milton (2006) first investigated the possibility for Navier's equation and found that the methodology cannot directly extend

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to elastodynamics due to the lack of form-invariant property. Some particular wave modes can still be examined, for example, the decoupled anti-plane shear waves (Guenneaua 2010) or bending waves in a thin plate (Farhat 2009). Brun (2009) studied the transformation techniques to design in-plane elastic transformation media, and later Hu (2010) proposed a general method to derive elastic transformation media. However, the corresponding set of material parameters appears either unsymmetric in the minors of elastic tensor or tonsorial value in mass density. Recently, Chang (2010) and Norris (2011) conclude that the transformation relations are not uniquely determined for elastodynamics. In other words, there are many possible choices in material specifications for a given transformation, and this inspires us to restudy the problem. In this paper, we introduce the transformation medium theory in elastodynamics under general curvilinear transformation and discuss their availability. Then we propose the affine transformation approach to derive the transformed fields and material parameters. Comparing to the previous work, the transformed mass density derived here is a scalar rather than a tensor and the corresponding elasticity tensor possesses full symmetries, which means that they are more practicable for application.

2. GENERAL CURVILINEAR TRANSFORMATION

2.1 General Curvilinear Coordinate systems

To begin with, we introduce the general curvilinear coordinate systems and list a few basic properties. Consider a general curvilinear coordinate ξ^1, ξ^2, ξ^3, which is defined by the rectangular Cartesian coordinates x_k , k =1, 2, 3 (as Fig. 1)

$$\begin{aligned} \xi^1 &= \xi^1(x_1, x_2, x_3), \\ \xi^2 &= \xi^2(x_1, x_2, x_3), \\ \xi^3 &= \xi^3(x_1, x_2, x_3). \end{aligned} \tag{1}$$

Suppose that $\mathbf{e}_k$ are the unit basis vectors of Cartesian coordinates, the position vector $r= x_1\mathbf{e}_1+x_2\mathbf{e}_2+x_3\mathbf{e}_3$ of a point in space can be expressed in the general coordinate k defined by the corresponding basis $\mathbf{g}_k$ as

$$\mathbf{r} = \xi^k \mathbf{g}_k = \xi^1 \mathbf{g}_1 + \xi^2 \mathbf{g}_2 + \xi^3 \mathbf{g}_3, \tag{2}$$

where

$$\mathbf{g}_k = \frac{\partial x_m}{\partial \xi^k} \mathbf{e}_m, \;\; k = 1, 2, 3. \tag{3}$$

The scalar numbers ξ^k in Eq. (2) are called components of the vector r with respect to the basis $\mathbf{g}_k$. Note that { $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ } must be a set of non-coplanar vectors, then any vector *F* can be written uniquely as

$$
\begin{aligned}
\mathbf{F} &= \bar{F}^k \mathbf{e}_k \\
&= F^k \mathbf{g}_k = F^1 \mathbf{g}_1 + F^2 \mathbf{g}_2 + F^3 \mathbf{g}_3 \\
&= F_k \mathbf{g}^k = F_1 \mathbf{g}^1 + F_2 \mathbf{g}^2 + F_3 \mathbf{g}^3 .
\end{aligned} \tag{4}
$$

Here the set {g^1, g^2, g^3} is the reciprocal basis defined by

$$
\mathbf{g}^k = \frac{\partial \xi^k}{\partial x_m} \mathbf{e}_m, \ \ k = 1, 2, 3. \tag{5}
$$

The components with lower and super scripts, $\bar{F}_k$ and $\bar{F}^k$, are referred to as the covariant and contravariant components along basis $\mathbf{g}_k$, respectively. We mention that the distinction of these two components is essential to our discussion, and we will show it later. The metric tensor of the coordinate is expressed as

$$
\mathbf{g} = \left[g_{ij} \right] = \mathbf{g}_i \cdot \mathbf{g}_j, \ \ i, j = 1, 2, 3. \tag{6}
$$

Since an infinitesimal vector displacement can be written as $\mathrm{d}\mathbf{r} = \mathrm{d}\xi^i \mathbf{g}_i$, it can be found that the infinitesimal arc length in terms of metric tensor is

$$
(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = g_{ij} d\xi^i d\xi^j , \tag{7}
$$

and the volume element cab be written as

$$
dV = \sqrt{g} d\xi^i d\xi^j d\xi^k , \tag{8}
$$

in which $g = |\, \boldsymbol{g} \,|$. It can further be shown that a second-order tensor E can be expressed in dyadic form as

$$
\mathbf{E} = \bar{E}^{ij} \mathbf{e}_i \otimes \mathbf{e}_j = E^{ij} \mathbf{g}_i \otimes \mathbf{g}_j = E_{ij} \mathbf{g}^i \otimes \mathbf{g}^j . \tag{9}
$$

Accordingly, $\bar{E}_{ij}$ and $\bar{E}^{ij}$ are covariant and contravariant components of E in curvilinear space. Similarly, a forth-order tensor can be represented by

$$
\begin{aligned}
\mathbf{D} &= \bar{D}^{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \\
&= D^{ijkl} \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k \otimes \mathbf{g}_l = D_{ijkl} \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \otimes \mathbf{g}^l .
\end{aligned} \tag{10}
$$

The double contraction of a fourth-order tensor with second-order tensor are defined by

$$\begin{aligned}\mathbf{D}:\mathbf{E} &= \left(D^{ijkl}\mathbf{g}_i\otimes\mathbf{g}_j\otimes\mathbf{g}_k\otimes\mathbf{g}_l\right):E_{kl}\mathbf{g}^k\otimes\mathbf{g}^l \\ &= \left(D^{ijkl}E_{kl}\right)\mathbf{g}_i\otimes\mathbf{g}_j.\end{aligned} \tag{11}$$

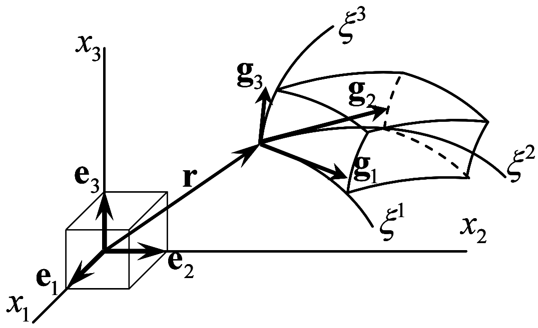

Fig. 1 Illustrations of general curvilinear coordinates

2.2 General Transformation in Elastodynamics

Next we discuss the transformation media theory in elastodynamics. Consider a coordinate transformation maps the original space (x_1, x_2, x_3) to the curvilinear space (ξ^1, ξ^2, ξ^3) as Fig. 1. The force balance in both systems can be written in the general form (assuming the body force vanishes)

$$\int_V \nabla\cdot\boldsymbol{\sigma}\,dV = \int_V \rho\ddot{\mathbf{u}}\,dV, \tag{12}$$

where σ and ρ denote the Cauchy stress tensor and mass density, respectively. Consider time harmonic cases, Eq. (12) can be expanded along the $\mathbf{e}_i$ basis, which yields

$$\left(\frac{\partial}{\partial x_i}\sigma_{ij}\right)\mathbf{e}_j = -\rho\omega^2 u_j\mathbf{e}_j, \tag{13}$$

or written in curvilinear coordinates with $\mathbf{g}_i$ basis as

$$\frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial\xi^i}\sqrt{g}\sigma^{ij}\mathbf{g}_j\right) = -\rho\omega^2 u^j\mathbf{g}_j. \tag{14}$$

Eq. (13) and (14) are called the Navier equations which govern the elastic stress wave propagation in medium. To expand Eq. (14), we will get

$$\left(\frac{\partial}{\partial\xi^i}\sigma^{ij} + \Gamma^i_{im}\sigma^{mj} + \Gamma^j_{im}\sigma^{im}\right)\mathbf{g}_j = -\rho\omega^2 u^j\mathbf{g}_j, \tag{15}$$

where Γ_{ij}^{k} are the Christoffel symbols of the second kind defined by

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{km} \left(\frac{\partial g_{mi}}{\partial \xi^{j}} + \frac{\partial g_{mj}}{\partial \xi^{i}} - \frac{\partial g_{ij}}{\partial \xi^{m}} \right). \tag{16}$$

According to the original concept in Pendry (2006), the transformation-based method is valid under the condition that governing equations is unchanged in their form after coordinate transformation. However, comparing Eq. (13) and (15), the form-invariant property cannot be observed since some additional terms are aroused.

3. AFFINE TRANSFORMATION AND FORM INVARIANCE OF GOVERNING EQUATIONS

Though transformation medium theory cannot directly extend to elastodynamics in fully general case, some special cases can be examined (Guenneaua 2010, Farhat 2009, Brun 2009). Here we propose an affine transformation to restudy the problem.

3.1 Curvilinearly Affine Transformation

Consider an curvilinearly affine transformation shown as Fig. 2 defined by

$$\begin{aligned} \xi'^{1} &= \alpha_1 \xi^1 + \beta_1, \\ \xi'^{2} &= \alpha_2 \xi^2 + \beta_2, \\ \xi'^{3} &= \alpha_3 \xi^3 + \beta_3. \end{aligned} \tag{17}$$

in which (ξ^1, ξ^2, ξ^3) represent an orthogonal curvilinear system. α_i and β_i are all constants. Eq. (17) can be regarded as the curvilinear space suffers a constant stretch, thus the corresponding basis vectors can be written simply as

$$\mathbf{g}'_k = \frac{1}{\alpha_K} \mathbf{g}_k, \; K = 1, 2, 3. \tag{18}$$

Note that the capital index denotes the same number as its lower case but without summation. Since the transformed basis is a set of orthogonal vectors, the remaining components of metric tensor can be written as

$$g'_{ij} = diag\left[\alpha_1^{-2} \quad \alpha_2^{-2} \quad \alpha_3^{-2} \right] g_{ij}. \tag{19}$$

and the volume ratio can be expressed in the form

$$\frac{dV'}{dV} = \frac{\sqrt{g'}}{\sqrt{g}}. \tag{20}$$

Under the coordinate stretching $\xi \to \xi'$, the governing equations should be written in the form

$$\frac{1}{\sqrt{g'}}\left(\frac{\partial}{\partial \xi'^i}\sqrt{g'}\sigma'^{ij}\mathbf{g}'_j\right) = -\rho'\omega^2 u'^j \mathbf{g}'_j. \tag{21}$$

Eq.(21) and (14) represent the same physical equations with respect to different basis. If we explicitly expand these two equations, it can be observed that no additional terms, or more precisely, no additional components of Christoffel symbols are aroused under the coordinate stretching and consequently the form invariance is held. This implies that the transformation medium theory can be applied in elastodynamics under such transformations.

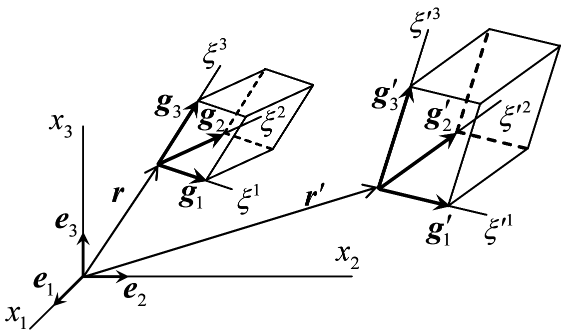

Figure 2 Illustrations of affine coordinate transformation.

3.2 Field Representation and Constitutive Equations

To further yield the material parameters, we must construct the constitutive relations in stretched coordinates. Due to the fact that each physical component of tensors is identical of these two coordinates, thus we can obtain the relations

$$\sqrt{g'}\sigma'^{ij} = \sqrt{g}\sigma^{ij}\alpha_I\alpha_J, \quad \sqrt{g'}\rho' = \sqrt{g}\rho, \quad u'^j = \alpha_J u^j. \tag{22}$$

Obviously, the transformed density can be directly rewritten as

$$\rho' = \frac{\sqrt{g}}{\sqrt{g'}}\rho. \tag{23}$$

The transformed displacement can be expressed in either covariant or contravariant components, and each is related to that in original space by

$$u'^j = \alpha_J u^j, \quad u'_j = \frac{1}{\alpha_J}u_j. \tag{24}$$

Following the notation of Eq. (11), it is convenient to chose the covariant components in the derivation of strain tensors. Under the assumption of small deformations, the strain tensor is defined by

$$\begin{aligned}\varepsilon'_{ij} &= \frac{1}{2}\left(\frac{\partial u'_i}{\partial \xi'^j} + \frac{\partial u'_j}{\partial \xi'^i}\right) - \Gamma'^k_{ij} u'_k \\ &= \frac{1}{2}\left(\frac{1}{\alpha_I \alpha_J}\frac{\partial u'_i}{\partial \xi'^j} + \frac{1}{\alpha_I \alpha_J}\frac{\partial u'_j}{\partial \xi'^i}\right) - \frac{\alpha_K}{\alpha_I \alpha_J}\Gamma^k_{ij}\frac{1}{\alpha_K}u_k \\ &= \frac{1}{\alpha_I \alpha_J}\varepsilon_{ij}.\end{aligned} \tag{25}$$

and thus the constitutive relation yields

$$\begin{aligned}\sigma'^{ij} &= \frac{\sqrt{g}}{\sqrt{g'}}\sigma^{ij}\alpha_I \alpha_J \\ &= \frac{\sqrt{g}}{\sqrt{g'}}\alpha_I \alpha_J C^{ijkl}\varepsilon_{kl} = \frac{\sqrt{g}}{\sqrt{g'}}\alpha_I \alpha_J C^{ijkl}\alpha_K \alpha_L \varepsilon'_{kl} = C'^{ijkl}\varepsilon'_{kl}\end{aligned} \tag{24}$$

where

$$C'^{ijkl} = \frac{\sqrt{g}}{\sqrt{g'}}\alpha_I \alpha_J \alpha_K \alpha_L C^{ijkl}. \tag{25}$$

Eq. (25) and Eq. (23) are called transformed elastic constants and mass density, respectively. It is seen that the mass density is a scalar rather than a diagonal tensor and the corresponding elasticity tensor possesses full symmetries. In particular, when the transformation function corresponds to a uniform stretching ($\alpha_1 = \alpha_2 = \alpha_3$) then the resulting material stiffness, Eq. (25), will become isotropic as long as the original elastic constant C^{ijkl} is isotropic as well.

3.3 Energy Conservation during Transformation

The kinetic energy and strain energy in the transformed space can be written as

$$K' = \frac{1}{2}\int_{V'} \rho' \dot{u}'_j \cdot \dot{u}'^j dV', \tag{26}$$

$$E' = \frac{1}{2}\int_{V'} \sigma'^{ij}\varepsilon'_{ij} dV'. \tag{27}$$

According to the relation Eq. (20), it can be shown that

$$K' = \frac{1}{2}\int_{V'} \rho' \dot{u}'_j \cdot \dot{u}'^j dV' = \frac{1}{2}\int_V \frac{\sqrt{g}}{\sqrt{g'}} \rho \frac{\dot{u}_j}{\alpha_J} \alpha_J \dot{u}^j \frac{\sqrt{g'}}{\sqrt{g}} dV$$
$$= \frac{1}{2}\int_V \rho \dot{u}_j \cdot \dot{u}^j dV = K. \quad (28)$$

Similarly, the strain energy can also be expressed as

$$E' = \frac{1}{2}\int_{V'} \sigma'^{ij} \varepsilon'_{ij} dV' = \frac{1}{2}\int_V \frac{\sqrt{g}}{\sqrt{g'}} \alpha_I \alpha_J \sigma^{ij} \frac{\varepsilon_{ij}}{\alpha_I \alpha_J} \frac{\sqrt{g}}{\sqrt{g'}} dV$$
$$= \frac{1}{2}\int_V \sigma^{ij} \varepsilon_{ij} dV = E. \quad (29)$$

Eq. (28) and (29) state that the kinetic energy and strain energy are conserved during the transformation. It seems that the energy conservation is a self-consistent result for transformation elastodynamics while the transformation function is determined, and thus it is inessential to consider additional constrains in choosing transformation functions (Hu 2010, Chang 2010).

CONCLUSION

In conclusion, we discuss the fesibility of transformation medium theory for elastodynamics and demonstrate that the Navier equations together with the constitutive relations could retain their form under affine transformation. Explicit derivation of transformed fields and material parameters in curvilinear coordinates are presented. The proposed materials in this text include a scalar density and a symmetrically anisotropic elastic tensor shown in Eq. (23) and (25), respectively.

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