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Three-Dimensional Analysis of Anisotropic Elastic Plates

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ABSTRACT

A three-dimensional analysis of transversely isotropic plates is performed, in which 5 independent material properties must be introduced, that is, two Young's moduli, two Poisson's ratios and a shear modulus . Numerical approaches are usually employed for three-dimensional analyses of anisotropic bodies. However, an analytical approach is adopted in the present investigation.

Fourier analysis approach is applied to the analysis of simply supported rectangular plates and it is shown that we can simply establish general solutions which involve 6 arbitrary constants, which can be determined completely by using traction boundary conditions on the top and bottom surfaces of plates. It follows from the solutions that the distinction between the surface tractions and the body force as lateral loads plays a key role in the three-dimensional bending analysis of plates.

1. INTRODUCTION

Many research works have been presented in three-dimensional analyses of isotropic and anisotropic elastic plates. Pagano (1970) constructs three-dimensional exact solutions for rectangular laminates. Srinivas and Rao (1970) present a three-dimensional, linear, small deflection theory of elasticity solution for the bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates. By using the exact three-dimensional theory of elasticity, Wittrick (1987) discusses buckling problems and free vibrations of a simply supported rectangular plate. Kaprielian et al. (1988) and Mian and Spencer (1998) deal with a three-dimensional analysis of functionally graded and laminated elastic plates. Tabakov (2005) discuss mathematical aspects of the three-dimensional analysis of laminated orthotropic plates. Chang and Tarn (2012) develop a state space approach for three-dimensional analysis of rectangular orthotropic elastic plates.

In those studies, however, only surface tractions are considered as lateral loads of plates and a body force along the thickness is usually ignored. In the present investigation, a body force in the direction of plate thickness is also taken into account. And it is discussed that the distinction between the surface tractions and the body force as lateral loads has great influence on mechanical behaviors of anisotropic plates.

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2. GOVERNING EQUATIONS OF ORTHOTROPIC PLATES

In the present investigation, orthotropic plates are dealt with. Especially, it is assumed that the materials of the plates are symmetric in two directions and there exists only one preferred direction. Namely, the materials are transversely isotropic. Governing equations for the transversely isotropic plates are derived in this section.

2.1. Constitutive Equations

A unidirectional fiber-reinforced plate is shown in Fig. 1 as an example of the transverse isotropic plates. This example has a preferred direction along the fibers, that is, the x-direction, and is symmetric in the y-z plane.

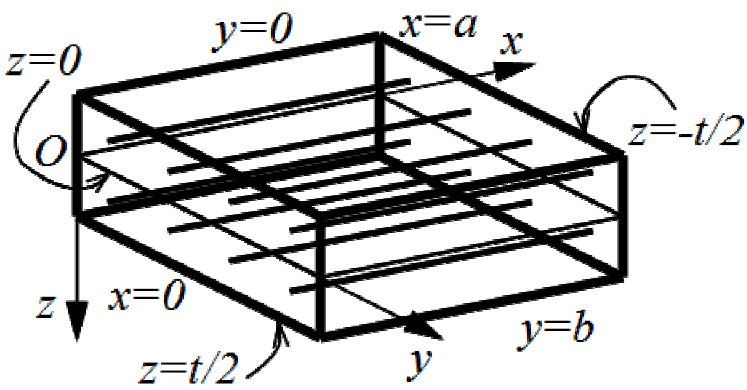

Fig. 1. A unidirectional fiber-reinforced plate.

Constitutive equations for transversely isotropic bodies are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} 1/E' & -\nu'/E' & -\nu'/E' & 0 & 0 & 0 \\ -\nu'/E' & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu'/E' & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G' & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G' \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}, \tag{1}$$

where $(\sigma_x, \sigma_y, \sigma_z)$ and $(\varepsilon_x, \varepsilon_y, \varepsilon_z)$ are normal stresses and strains; $(\tau_{xy}, \tau_{yz}, \tau_{zx})$ and $(\gamma_{xy}, \gamma_{yz}, \gamma_{zx})$ are shear stresses and strains; E is a Young's modulus in the x-direction and E' in the y- or z-directions; $G \equiv E/2(1+\nu)$ and ν are a shear modulus and a Poisson's ratio, respectively, in the y-z plane, and G' and ν' in the x-y or the z-x planes. The inverse relations of Eq. (1) are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} = \frac{E}{c_1} \begin{bmatrix} \dfrac{1-\nu}{\alpha} & \nu' & \nu' \\ \nu' & \dfrac{1-\alpha\nu'^2}{1+\nu} & \dfrac{\nu+\alpha\nu'^2}{1+\nu} \\ \nu' & \dfrac{\nu+\alpha\nu'^2}{1+\nu} & \dfrac{1-\alpha\nu'^2}{1+\nu} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} \quad , \quad \begin{Bmatrix} \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} G' & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G' \end{bmatrix} \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}, \qquad (2)$$

where $\alpha \equiv E/E'$ and $c_1 \equiv 1-\nu-2\alpha\nu'^2$.

2.2. Governing Equations

Substituting Eq. (2) into the 3-D equilibrium equations, in view of the strain-displacement relations, we have the following equations:

$$\begin{cases} \{(1-\nu)\dfrac{\partial^2}{\partial x^2} + \beta c_1 (\dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2})\}U + c_2 \dfrac{\partial}{\partial x}(\dfrac{\partial V}{\partial y} + \dfrac{\partial W}{\partial z}) = 0 \\ \{\beta c_1 \dfrac{\partial^2}{\partial x^2} + \dfrac{\alpha(1-\alpha\nu'^2)}{1+\nu}\dfrac{\partial^2}{\partial y^2} + \dfrac{\alpha c_1}{2(1+\nu)}\dfrac{\partial^2}{\partial z^2})\}V + c_2 \dfrac{\partial^2 U}{\partial x \partial y} + \dfrac{\alpha}{2}\dfrac{\partial^2 W}{\partial y \partial z} = 0 \\ \{\beta c_1 \dfrac{\partial^2}{\partial x^2} + \dfrac{\alpha c_1}{2(1+\nu)}\dfrac{\partial^2}{\partial y^2} + \dfrac{\alpha(1-\alpha\nu'^2)}{1+\nu}\dfrac{\partial^2}{\partial z^2}\}W + c_2 \dfrac{\partial^2 U}{\partial x \partial z} + \dfrac{\alpha}{2}\dfrac{\partial^2 V}{\partial y \partial z} + \alpha c_1 \dfrac{\overline{p}_0}{Et} = 0 \end{cases} \quad , \qquad (3)$$

where (U,V,W) are displacements in the x-, the y-, and the z-directions, t is a thickness of plates, $\beta \equiv G'/E'$ and $c_2 \equiv \alpha\nu' + \beta c_1$. In Eq. (3), only the body force in the z-direction $\overline{p}_0/t$, which is constant, is considered and the body forces in the other two directions are ignored. The body force considered here corresponds to, for example, a dead load of plates. Note that the body force is usually ignored in the previous investigations. Equation (3) governs the bending behavior of transversely isotropic plates.

3. FOURIER ANALYSIS

3.1. General Solutions

Since fully simply-supported plates are dealt with in this paper, the displacements in the three directions can be expressed as the following trigonometric series:

$$\begin{cases} U = \sum_k \sum_j U_{jk}(z)\cos\lambda_j x \sin\mu_k y \\ V = \sum_k \sum_j V_{jk}(z)\sin\lambda_j x \cos\mu_k y \\ W = \sum_k \sum_j W_{jk}(z)\sin\lambda_j x \sin\mu_k y \end{cases} \quad ; \quad \lambda_j \equiv \frac{j\pi}{a}, \quad \mu_k \equiv \frac{k\pi}{b}. \qquad (4)$$

The displacement field (4) satisfies the following geometrical boundary conditions a priori.

$$\begin{cases} U(x,0,z) = U(x,b,z) = 0\ ,\ V(0,y,z) = V(a,y,z) = 0 \\ W(0,y,z) = W(a,y,z) = W(x,0,z) = W(x,b,z) = 0 \end{cases} . \tag{5}$$

Substituting Eq. (4) into Eq. (3), we have the simultaneous ordinary differential equations for determining $U_{jk}(z)$, $V_{jk}(z)$, and $W_{jk}(z)$. A matrix expression of the differential equations is

$$\begin{bmatrix} \beta c_1(d^2-\mu_k{}^2)-(1-\nu)\lambda_j{}^2 & -c_2\lambda_j\mu_k & c_2\lambda_j d \\ -2(1+\nu)c_2\lambda_j\mu_k & \alpha\{c_1(d^2-{}_0\kappa_{jk}{}^2)-(1+\nu)\mu_k{}^2\} & \alpha(1+\nu)\mu_k d \\ -2(1+\nu)c_2\lambda_j d & -\alpha(1+\nu)\mu_k d & \alpha\{c_1(d^2-{}_0\kappa_{jk}{}^2)+(1+\nu)d^2\} \end{bmatrix}$$
$$\times\begin{Bmatrix} U_{jk} \\ V_{jk} \\ W_{jk} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -2\alpha(1+\nu)c_1\dfrac{\bar{p}_0}{Et} \end{Bmatrix}\quad ;\quad d \equiv \frac{d}{dz}\quad ,\quad {}_0\kappa_{jk}{}^2 \equiv \frac{2\beta}{\alpha}(1+\nu)\lambda_j{}^2+\mu_k{}^2. \tag{6}$$

In order to obtain complementary solutions of Eq. (6), a characteristic equation is derived from the determinant of the coefficient matrix as

$$\begin{vmatrix} \beta c_1(d^2-\mu_k{}^2)-(1-\nu)\lambda_j{}^2 & -c_2\lambda_j\mu_k & c_2\lambda_j d \\ -2(1+\nu)c_2\lambda_j\mu_k & \alpha\{c_1(d^2-{}_0\kappa_{jk}{}^2)-(1+\nu)\mu_k{}^2\} & \alpha(1+\nu)\mu_k d \\ -2(1+\nu)c_2\lambda_j d & -\alpha(1+\nu)\mu_k d & \alpha\{c_1(d^2-{}_0\kappa_{jk}{}^2)+(1+\nu)d^2\} \end{vmatrix}$$
$$= \alpha^2 c_1{}^2(d^2-{}_0\kappa_{jk}{}^2)[\beta(1+\nu+c_1)(d^2-\mu_k{}^2)^2 \tag{7}$$
$$-2\{1-2\beta(1+\nu)\nu'\}\lambda_j{}^2(d^2-\mu_k{}^2)+\frac{2\beta}{\alpha}(1-\nu^2)\lambda_j{}^4]=0.$$

The roots of the characteristic equation are

$$d = \pm{}_0\kappa_{jk}\ ,\ \pm{}_1\kappa_{jk} \equiv \sqrt{K_1\lambda_j{}^2+\mu_k{}^2}\ ,\ \pm{}_2\kappa_{jk} \equiv \sqrt{K_2\lambda_j{}^2+\mu_k{}^2}$$
$$;\quad K_1\ ,\ K_2 \equiv \frac{1}{\beta(1+\nu+c_1)}\{1-2\beta(1+\nu)\nu' \pm \sqrt{1-4\frac{\beta}{\alpha}(1+\nu)c_2}\}. \tag{8}$$

From Eq. (8), we can obtain a system of elementary solutions:

$$\sinh{}_0\kappa_{jk}z\ ,\ \cosh{}_0\kappa_{jk}z\ ,\ \sinh{}_1\kappa_{jk}z\ ,\ \cosh{}_1\kappa_{jk}z\ ,\ \sinh{}_2\kappa_{jk}z\ ,\ \cosh{}_2\kappa_{jk}z. \tag{9}$$

At this stage, we need to derive general solutions of the homogeneous equation of Eq. (6). In order to do that, the following set of solutions for ${}_0\kappa_{jk}$ is selected here:

$$U_{jk}(z) = A_{jk}^{(1)} \cosh {}_0\kappa_{jk} z + B_{jk}^{(1)} \sinh {}_0\kappa_{jk} z \quad , \quad V_{jk}(z) = A_{jk}^{(2)} \cosh {}_0\kappa_{jk} z + B_{jk}^{(2)} \sinh {}_0\kappa_{jk} z \quad ,$$
$$W_{jk}(z) = A_{jk}^{(3)} \sinh {}_0\kappa_{jk} z + B_{jk}^{(3)} \cosh {}_0\kappa_{jk} z. \tag{10}$$

where $A_{jk}^{(i)}(z)$ and $B_{jk}^{(i)}(z)$ $(i = 1 \sim 3)$ are arbitrary constants. Substituting Eq. (10) into the homogeneous equation of Eq. (6), we have the following relation among the arbitrary constants $A_{jk}^{(i)}(z)$ and $B_{jk}^{(i)}(z)$:

$$\begin{bmatrix} \{\frac{2\beta^2 c_1}{\alpha}(1+\nu)-1+\nu\}\lambda_j & -c_2\mu_k & c_2\,{}_0\kappa_{jk} \\ -2c_2\lambda_j & -\alpha\mu_k & \alpha\,{}_0\kappa_{jk} \\ -2c_2\lambda_j & -\alpha\mu_k & \alpha\,{}_0\kappa_{jk} \end{bmatrix} \begin{Bmatrix} A_{jk}^{(1)} \\ A_{jk}^{(2)} \\ A_{jk}^{(3)} \end{Bmatrix} \ or \ \begin{Bmatrix} B_{jk}^{(1)} \\ B_{jk}^{(2)} \\ B_{jk}^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{11}$$

Elementary row operations on the coefficient matrix gives us the following row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -{}_0\kappa_{jk}/\mu_k \\ 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

Therefore, we can find that the rank of the matrix is 2 and obtain arbitrary constant vectors for the solutions (10) as

$$\begin{Bmatrix} A_{jk}^{(1)} \\ A_{jk}^{(2)} \\ A_{jk}^{(3)} \end{Bmatrix} = A_{jk} \begin{Bmatrix} 0 \\ {}_0\kappa_{jk}/\mu_k \\ 1 \end{Bmatrix} \quad , \quad \begin{Bmatrix} B_{jk}^{(1)} \\ B_{jk}^{(2)} \\ B_{jk}^{(3)} \end{Bmatrix} = B_{jk} \begin{Bmatrix} 0 \\ {}_0\kappa_{jk}/\mu_k \\ 1 \end{Bmatrix}. \tag{13}$$

In the similar manner to the above process, we obtain constant vectors for solutions corresponding to ${}_1\kappa_{jk}$ and ${}_2\kappa_{jk}$. Thus we can derive general solutions of Eq. (6) as

$$\begin{aligned} U_{jk}(z) = {} & \frac{c_2 K_1}{1-\nu-\beta c_1 K_1} \cdot \frac{\lambda_j}{{}_1\kappa_{jk}} (C_{jk} \cosh {}_1\kappa_{jk} z + D_{jk} \sinh {}_1\kappa_{jk} z) \\ & + \frac{c_2 K_2}{1-\nu-\beta c_1 K_2} \cdot \frac{\lambda_j}{{}_2\kappa_{jk}} (E_{jk} \cosh {}_2\kappa_{jk} z + F_{jk} \sinh {}_2\kappa_{jk} z), \end{aligned} \tag{14}$$

$$V_{jk}(z)=\frac{{}_0\kappa_{jk}}{\mu_k}(A_{jk}\cosh{}_0\kappa_{jk}z+B_{jk}\sinh{}_0\kappa_{jk}z)$$
$$+\frac{\mu_k}{{}_1\kappa_{jk}}(C_{jk}\cosh{}_1\kappa_{jk}z+D_{jk}\sinh{}_1\kappa_{jk}z)+\frac{\mu_k}{{}_2\kappa_{jk}}(E_{jk}\cosh{}_2\kappa_{jk}z+F_{jk}\sinh{}_2\kappa_{jk}z), \quad (15)$$
$$W_{jk}(z)=A_{jk}\sinh{}_0\kappa_{jk}z+B_{jk}\cosh{}_0\kappa_{jk}z$$
$$+C_{jk}\sinh{}_1\kappa_{jk}z+D_{jk}\cosh{}_1\kappa_{jk}z+E_{jk}\sinh{}_2\kappa_{jk}z+F_{jk}\cosh{}_2\kappa_{jk}z+\frac{\bar{P}_{jk}^{(0)}}{Gt\tilde{\kappa}_{jk}{}^2}. \quad (16)$$

where a particular solution for $W_{jk}(z)$ is included and A_{jk} to F_{jk} are arbitrary constants.

3.2. Traction Boundary Conditions

The 6 arbitrary constants A_{jk} to F_{jk} are determined by applying the traction boundary conditions on the top and bottom surfaces of plates. Those traction boundary conditions are given by

$$\sigma_z(x,y,-\frac{t}{2})=-\bar{p}_1(x,y)\ ,\ \sigma_z(x,y,\frac{t}{2})=\bar{p}_2(x,y)\ ,\ \tau_{xz}(x,y,\pm\frac{t}{2})=\tau_{yz}(x,y,\pm\frac{t}{2})=0. \quad (17)$$

By using these 6 boundary conditions, we can determine the 6 arbitrary constants A_{jk} to F_{jk} as

$$A_{jk}=-\frac{1}{\tilde{D}_1}(K_1^*-K_2^*)sh_{jk}^{(1)}\cdot sh_{jk}^{(2)}\frac{\bar{P}_{jk}^{(1)}-\bar{P}_{jk}^{(2)}}{2G}, \quad (18)$$
$$B_{jk}=\frac{1}{\tilde{D}_2}[\{(2-K_2^*)\frac{({}_1\kappa_{jk}^*)^2}{{}_1\kappa_{jk}}sh_{jk}^{(1)}\cdot ch_{jk}^{(2)}-(2-K_1^*)\frac{({}_2\kappa_{jk}^*)^2}{{}_2\kappa_{jk}}ch_{jk}^{(1)}\cdot sh_{jk}^{(2)}\}\frac{\bar{P}_{jk}^{(0)}}{Gt\,{}_0\kappa_{jk}{}^2}$$
$$+(K_1^*-K_2^*)ch_{jk}^{(1)}\cdot ch_{jk}^{(2)}\frac{\bar{P}_{jk}^{(1)}+\bar{P}_{jk}^{(2)}}{2G}], \quad (19)$$
$$C_{jk}=\frac{1}{\tilde{D}_1}\{2-K_2^*(1+\frac{{}_0\kappa_{jk}{}^2}{\mu_k{}^2})\}sh_{jk}^{(0)}\cdot sh_{jk}^{(2)}\frac{\bar{P}_{jk}^{(1)}-\bar{P}_{jk}^{(2)}}{4G}, \quad (20)$$
$$D_{jk}=\frac{1}{\tilde{D}_2}[\{\frac{{}_0\kappa_{jk}{}^2}{\mu_k{}^2}\cdot\frac{({}_2\kappa_{jk}^*)^2}{{}_2\kappa_{jk}}ch_{jk}^{(0)}\cdot sh_{jk}^{(2)}-(2-K_2^*)\,{}_0\kappa_{jk}sh_{jk}^{(0)}\cdot ch_{jk}^{(2)}\}\frac{\bar{P}_{jk}^{(0)}}{Gt\,{}_0\kappa_{jk}{}^2}$$
$$-\{2-K_2^*(1+\frac{{}_0\kappa_{jk}{}^2}{\mu_k{}^2})\}ch_{jk}^{(0)}\cdot ch_{jk}^{(2)}\frac{\bar{P}_{jk}^{(1)}+\bar{P}_{jk}^{(2)}}{4G}], \quad (21)$$
$$E_{jk}=-\frac{1}{\tilde{D}_1}\{2-K_1^*(1+\frac{{}_0\kappa_{jk}{}^2}{\mu_k{}^2})\}sh_{jk}^{(0)}\cdot sh_{jk}^{(1)}\frac{\bar{P}_{jk}^{(1)}-\bar{P}_{jk}^{(2)}}{4G}, \quad (22)$$

$$F_{jk} = -\frac{1}{\tilde{D}_2}[\{\frac{{}_0\kappa_{jk}{}^2}{\mu_k{}^2}\cdot\frac{({}_1\kappa^*_{jk})^2}{{}_1\kappa_{jk}} ch^{(0)}_{jk}\cdot sh^{(1)}_{jk} - (2-K^*_1)\,{}_0\kappa_{jk} sh^{(0)}_{jk}\cdot ch^{(1)}_{jk}\}\frac{\overline{P}^{(0)}_{jk}}{Gt\,{}_0\kappa_{jk}{}^2}$$
$$-\{2-K^*_1(1+\frac{{}_0\kappa_{jk}{}^2}{\mu_k{}^2})\}ch^{(0)}_{jk}\cdot ch^{(1)}_{jk}\frac{\overline{P}^{(1)}_{jk}+\overline{P}^{(2)}_{jk}}{4G}]. \tag{23}$$

where

$$sh^{(0)}_{jk} \equiv \sinh\frac{{}_0\kappa_{jk}t}{2}\ ,\ ch^{(0)}_{jk} \equiv \cosh\frac{{}_0\kappa_{jk}t}{2}\ ,\ sh^{(1)}_{jk} \equiv \sinh\frac{{}_1\kappa_{jk}t}{2}\ ,\ ch^{(1)}_{jk} \equiv \cosh\frac{{}_1\kappa_{jk}t}{2}\ ,$$
$$sh^{(2)}_{jk} \equiv \sinh\frac{{}_2\kappa_{jk}t}{2}\quad ,\quad ch^{(2)}_{jk} \equiv \cosh\frac{{}_2\kappa_{jk}t}{2}\quad , \tag{24}$$

$${}_\rho\kappa^*_{jk} \equiv \sqrt{\frac{\beta}{\alpha}(1+\nu)K^*_1\lambda_j{}^2 + \mu_k{}^2}\quad ;\quad K^*_\rho \equiv 1+\frac{c_2K_\rho}{1-\nu-\beta c_1 K_\rho}\quad (\rho = 1,2), \tag{25}$$

$$\tilde{D}_1 \equiv 2(K^*_1-K^*_2)\,{}_0\kappa_{jk} ch^{(0)}_{jk}\cdot sh^{(1)}_{jk}\cdot sh^{(2)}_{jk} - \{2-K^*_2(1+\frac{{}_0\kappa_{jk}{}^2}{\mu_k{}^2})\}\frac{({}_1\kappa^*_{jk})^2}{{}_1\kappa_{jk}} sh^{(0)}_{jk}\cdot ch^{(1)}_{jk}\cdot sh^{(2)}_{jk}$$
$$+\{2-K^*_1(1+\frac{{}_0\kappa_{jk}{}^2}{\mu_k{}^2})\}\frac{({}_2\kappa^*_{jk})^2}{{}_2\kappa_{jk}} sh^{(0)}_{jk}\cdot sh^{(1)}_{jk}\cdot ch^{(2)}_{jk}, \tag{26}$$

$$\tilde{D}_2 \equiv 2(K^*_1-K^*_2)\,{}_0\kappa_{jk} sh^{(0)}_{jk}\cdot ch^{(1)}_{jk}\cdot ch^{(2)}_{jk} - \{2-K^*_2(1+\frac{{}_0\kappa_{jk}{}^2}{\mu_k{}^2})\}\frac{({}_1\kappa^*_{jk})^2}{{}_1\kappa_{jk}} ch^{(0)}_{jk}\cdot sh^{(1)}_{jk}\cdot ch^{(2)}_{jk}$$
$$+\{2-K^*_1(1+\frac{{}_0\kappa_{jk}{}^2}{\mu_k{}^2})\}\frac{({}_2\kappa^*_{jk})^2}{{}_2\kappa_{jk}} ch^{(0)}_{jk}\cdot ch^{(1)}_{jk}\cdot sh^{(2)}_{jk}. \tag{27}$$

In addition, the lateral loads of plates $\overline{p}_i(x,y)\ (i = 0 \sim 2)$, which include the body force, are assumed to be expanded into Fourier double series as

$$\overline{p}_i(x,y) = \sum_j\sum_k \overline{P}^{(i)}_{jk}\sin\lambda_j x\sin\mu_k y\quad ;\quad \overline{P}^{(i)}_{jk} = \frac{4}{ab}\int_0^b\int_0^a \overline{p}_i(x,y)\sin\lambda_j x\sin\mu_k y dxdy. \tag{28}$$

Note that influence of the body force is included in Eqs. (19), (21), and (23).

4. INFLUENCE OF BODY FORCE

As mentioned before, the effect of the body force in the lateral direction has been almost ignored in the previous investigations. However, it is apparent from Eqs. (18) to (23) that the body force deeply affect bending behavior of plates. Although the surface tractions on the top and bottom surfaces of plates are included in all the 6 constants A_{jk} to F_{jk}, the body force is not included in A_{jk}, C_{jk}, and E_{jk}. Therefore, when we consider only the body force as a lateral load of plates, the deflection must be symmetric with respect to the mid-surface of plates and the in-plane displacements anti-symmetric. On the other hand, when we ignore the body force, we cannot distinguish the symmetry and the anti-symmetry in the bending behavior of plates. Thus it is clear that the distinction between the surface tractions and the body force as lateral loads of plates plays a key role in the three-dimensional bending analysis of plates.

Numerical examples of transversely isotropic plates will be presented at the conference. In this section, examples of isotropic plates in Suetake (2007) are presented in order to show differences among mechanical behaviors of plates under the surface traction and the body force. Such numerical examples are shown in Fig. 2 and Fig. 3, in which solid lines indicate results of the body force case and broken lines results of the surface traction case. In both figures, the vertical axis indicates a non-dimensional coordinate along the thickness, $\zeta \equiv z/t$. Distribution of an in-plane displacement along the thickness at $(x, y) = (a, b/2)$ is shown in Fig. 2, in which a non-dimensional in-plane displacement U/a is depicted against ζ. On the other hand, distribution of a transverse shear stress along the thickness at $(x, y) = (a/4, b/4)$ is shown in Fig. 3, in which a non-dimensional transverse shear stress $\tau^* \equiv \{(1-2\nu)/2G\}\tau_{zx}$ is depicted against ζ.

It can be seen from both figures that differences between the body force and the surface traction cases are outstanding even in bending behaviors of isotropic plates. Therefore we can expect that more outstanding differences are found in the analyses of anisotropic plates. In addition, it is expected that a drilling rotation of plates, $\partial U/\partial y - \partial V/\partial x$, does not become zero under only lateral loads in the analyses of anisotropic plates.

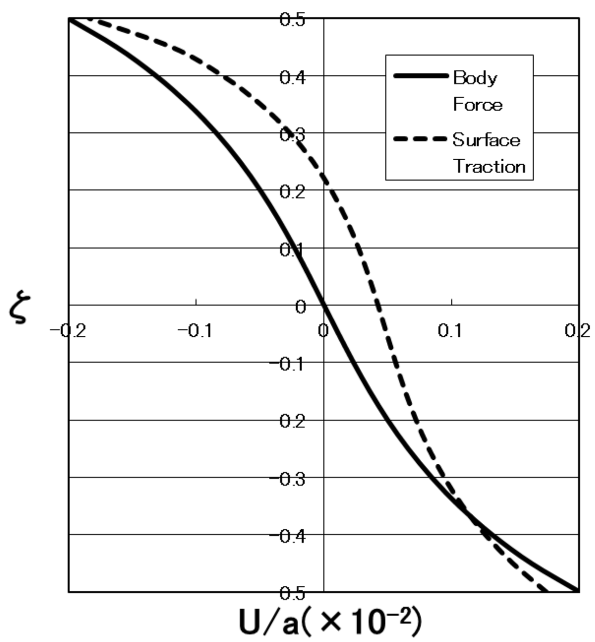

Fig. 2. In-plane displacement distribution

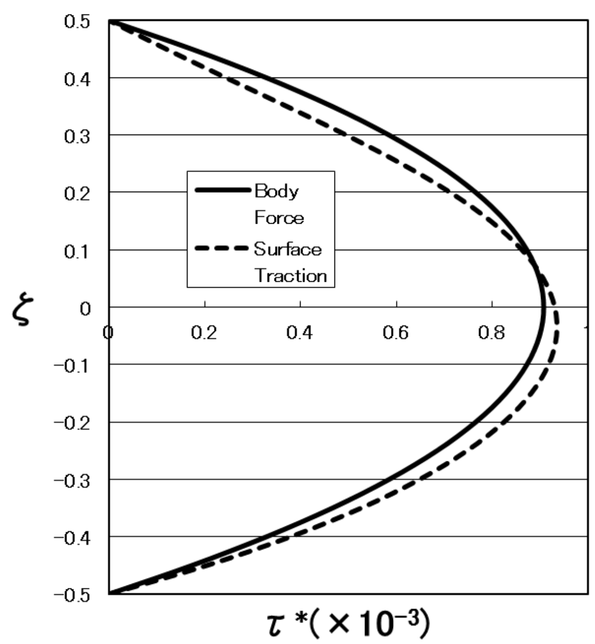

Fig. 3. Transverse shear stress distribution

CONCLUSION

In the present investigation, a three-dimensional analysis of anisotropic elastic plates is dealt with, in which material of plates is assumed to be transversely isotropic. Through the Fourier analysis, an exact solution is derived for simply supported rectangular plates with any thickness. Especially, the body force effect is taken into account in the present solution. It follows from the solutions derived here that the distinction between the surface tractions and the body force as lateral loads has great influence on mechanical behaviors of plates.

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