The 2012 World Congress on
Advances in Civil, Environmental, and Materials Research (ACEM' 12)
Seoul, Korea, August 26-30, 2012

Symmetry-Breaking Deformation Mode in Radially-Pressurized Complete Spherical Shells

Motohiro Sato[1)], *Takafumi Sekizawa[2)] and Yoshiteru Konishi[2)]

[1)] *Division of Engineering and Policy for Sustainable Environment, Faculty of Engineering, Hokkaido University, Kita-13, Nishi-8, Kita-ku, Sapporo, Japan*
[2)] *Division of Engineering and Policy for Sustainable Environment, Graduate School of Engineering, Hokkaido University, Kita-13, Nishi-8, Kita-ku, Sapporo, Japan*
[1)] tayu@eng.hokudai.ac.jp

ABSTRACT

The critical buckling characteristics of hydrostatically pressurized complete spherical shells filled with an elastic medium are demonstrated. A model based on small deflection thin shell theory, the equations of which are solved in conjunction with variational principles, is presented. Axisymmetric and inextensional assumptions are not used initially in the exact formulation and the elastic medium is modeled as a Winkler foundation. Simplified approximations based on a Rayleigh-Ritz approach are also introduced for the critical buckling pressure. The present exact formulation can be readily extended to apply to more general cases of non-axisymmetric buckling problems and the approximate method can be extended to the post-buckling range.

1. INTRODUCTION

The analytical study about the structural behavior of shallow and complete spherical shells is of great importance not only in the fields of civil, mechanical and aeronautical engineering but also in nanoscience and biomechanics. Notable examples include pressure vessels, spherical honeycombs, carbon onions, spherical viruses and so on. From the background described above, the buckling properties of hydrostatically pressurized complete spherical shells filled with an elastic medium is demonstrated currently. The paper concludes by outlining how the present study can be extended into the nonlinear range and for investigating of non-axisymmetric cases that include the properties of carbon onions.

2. COMPLETE SPHERICAL SHELL MODEL

The critical pitchfork bifurcation phenomenon of the hydrostatically pressurized complete spherical shell is investigated, as shown in Fig. 1. The spherical shell is constructed from a homogeneous and isotropic linear elastic material with Young's modulus E and Poisson's ratio v, which is filled with an elastic material that is modeled as a Winkler foundation, i.e. with uncoupled springs in the radial direction and a constant foundation modulus k_f. For a complete spherical shell with radius a and thickness h, spherical angular coordinates in the latitude and meridian directions θ and ϕ are used. Displacement functions: u, v and w are in the θ, ϕ and the radial directions, respectively.

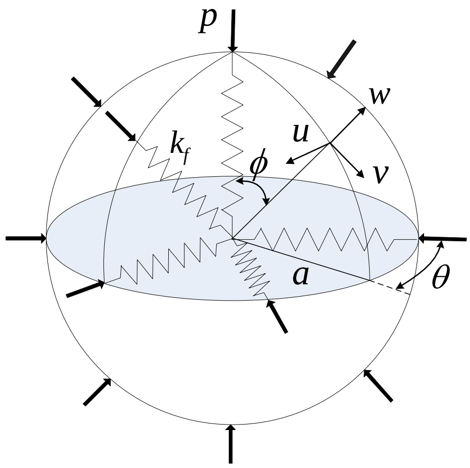

Fig.1Hydrostatically pressurized spherical shell filled with an elastic medium

3. FORMULATION

3.1. EXACT APPROACH

The following analysis is based on classical small deformation theory of thin shells. The total potential energy V is expressed by the sum of the strain energy and work done by external load as Eq.(1).

$$V = U_M + U_B + U_F + \Omega \tag{1}$$

In which U_M is the membrane strain energy term, U_B is the bending component, U_F is the strain energy term due to a Winkler foundation and Ω is the potential energy of the applied pressure.When uniform hydrostatic pressure p is acting on the spherical shell, only the inward radial static displacement w_0 can occur in the pre-buckling state. Hence, the stress-strain relationship of the spherical shell is assumed to be linear up to the point of instability. In the fundamental pre-buckling state, the potential energies $V^{(0)}$ are given by Eq.(2).

$$V^{(0)} = U_M^{(0)} + U_B^{(0)} + U_F^{(0)} + \Omega^{(0)} = \int_\phi \int_\theta F^{(0)} d\phi d\theta \tag{2}$$

For an equilibrium state, the first variation of the total potential energy V must equal zero. This condition gives the following static displacement under uniform hydrostatic pressure:

$$w_0 = -\frac{a^2(1-\nu)}{2Eh + a^2 k_f (1-\nu)} p \tag{3}$$

The second variation of the total potential energy becomes

$$\delta^2 U = \delta^2 U_M + \delta^2 U_B + \delta^2 U_F = \int_\phi \int_\theta F d\phi d\theta \tag{4}$$

According to the Trefftz criterion, the buckling equations can be obtained by introducing F in Eq.(4) into the Euler-Lagrange equations with the calculus of variations. The Euler-Lagrange equations in this case which is applied substitution of a solution are as follows:

$$\begin{aligned}
&a_u^{(i)} \frac{\partial^i \bar{u}}{\partial \phi^i} + b_u^{(i)} \frac{\partial^i \bar{v}}{\partial \phi^i} + c_u^{(i)} \frac{\partial^i \bar{w}}{\partial \phi^i} - \frac{w_0}{a}(1+v)(\bar{u}\sin\phi + n\bar{w}) = 0 \\
&a_v^{(i)} \frac{\partial^i \bar{u}}{\partial \phi^i} + b_v^{(i)} \frac{\partial^i \bar{v}}{\partial \phi^i} + c_v^{(i)} \frac{\partial^i \bar{w}}{\partial \phi^i} - \frac{w_0}{a}(1+v)\sin\phi(\bar{v} - \frac{\partial \bar{w}}{\partial \phi}) = 0 \\
&a_w^{(i)} \frac{\partial^i \bar{u}}{\partial \phi^i} + b_w^{(i)} \frac{\partial^i \bar{v}}{\partial \phi^i} + c_w^{(i)} \frac{\partial^i \bar{w}}{\partial \phi^i} + \frac{a^2 k_f}{C} \bar{w}\sin\phi + \frac{w_0}{a}(1+v)\sin\phi \\
&\times \left(\frac{\partial \bar{v}}{\partial \phi} + \bar{v}\cot\phi + \frac{n\bar{u}}{\sin\phi} - \frac{\partial^2 \bar{w}}{\partial \phi^2} - \frac{\partial \bar{w}}{\partial \phi}\cot\phi + \frac{n^2}{\sin^2\phi}\bar{w} \right) = 0
\end{aligned} \tag{5}$$

Eq.(5) are the governing equations to be solved. Now, the solution of ψ and w are assumed to be in the form:

$$\psi = \sum_{m=0}^{\infty} A_m P_m(\cos\phi),\ w = \sum_{m=0}^{\infty} B_m P_m(\cos\phi), \tag{6}$$

where A_m and B_m are the constant deformation amplitudes, and the spherical harmonic $P_m(\cos\phi)$ is a series of Legendre functions of degree m.
Hence, homogeneous linear equations for A_m and B_m are obtained for all m as follows:

$$\begin{bmatrix} c_m^{(11)} & c_m^{(12)} \\ c_m^{(21)} & c_m^{(22)} \end{bmatrix} \begin{pmatrix} A_m \\ B_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7}$$

In which

$$c_m^{(11)} = (1+k)(\lambda_m + 1 + \nu) + \frac{w_0}{a}(1+\nu)$$

$$c_m^{(12)} = (1+k)(1+\nu)(\lambda_m + 2) + k(\lambda_m^2 + 2k\lambda_m) + \frac{w_0}{a}(1+\nu)(1+\lambda_m)$$

$$c_m^{(21)} = (1+k)(1+\nu) + k\lambda_m + \frac{w_0}{a}(1+\nu)$$

$$c_m^{(12)} = 2(1+k)(1+\nu)(\lambda_m + 2) + k\lambda_m(\lambda_m + 3 + \nu) + \frac{a^2 k_f}{C} + \frac{w_0}{a}(1+\nu)(\lambda_m + 2) \quad (8)$$

This is a standard eigenvalue problem for which nontrivial values are obtained when the coefficient matrix in Eq.(7) becomes singular. The corresponding pressure p is thus calculated from the determinantal equations:

$$\det\begin{bmatrix} c_m^{(11)} & c_m^{(12)} \\ c_m^{(21)} & c_m^{(22)} \end{bmatrix} = 0 \quad (9)$$

By solving Eq.(9), the pressure p with the corresponding mode m for several combinations of geometric and material constants can be obtained as an eigenvalue. For the particular values of the constants, the minimum value of the eigenvalues is the critical buckling pressure p_{cr}. It should be noted that the above equation is independent of the mode number m. This shows that only axisymmetric modes can occur in this problem despite the inclusion of the Winkler foundation term.

3.2 SIMPLIFIED APPROACH

Next, we consider the simplified formulation using the Rayleigh-Ritz approach. Now, an inextensional and axisymmetric buckling deformation of the shell are assumed with shear and torsional strains being neglected for simplicity. Hence, u, $\varepsilon_{\phi\theta}$, and $\chi_{\phi\theta}$ are assumed to be zero.
The stability criterion

$$\frac{\partial(\delta^2 V)}{\partial \tilde{v}_m} = 0 \quad (10)$$

gives the following critical pressure:

$$p_{cr} = \frac{2Eh + k_f a^2(1-\nu)}{aEh(m-1)^2(m+2)^2}\left\{\frac{2Eh + (m-1)(m+2)}{1+\nu} + \frac{D(m-1)^2(m+2)^2\left[m(m+1) - 1 + \nu\right]}{a^2} + a^2 k_f m(m+1)\right\} \quad (11)$$

4. ANALYTICAL RESULTS AND DISCUSSION

Fig.2 shows the variation of the nondimensionalized critical buckling pressure p_{cr}/p_0, in which p_0 is the critical pressure for the empty complete spherical shell, which is given by

$$p_0 = \frac{2E}{\sqrt{3(1-\nu^2)}}\left(\frac{h}{a}\right)^2 \tag{12}$$

The solid lines are the exact values obtained from Eq.(9) and the dotted lines correspond to Eq.(11) from the Rayleigh-Ritz analysis. As can be easily seen in this figure, the value of p_{cr}/p_0 increases with increasing stiffness ratio ak_f/E and the exact and simplified p_{cr} curve agree with each other.
Fig.3 illustrates the buckling eigenmodes for the empty complete spherical shell of $a/h=50$. The characteristic wavy-shaped axisymmetric buckling deformation with the mode number $m=18$ can be found in this case.

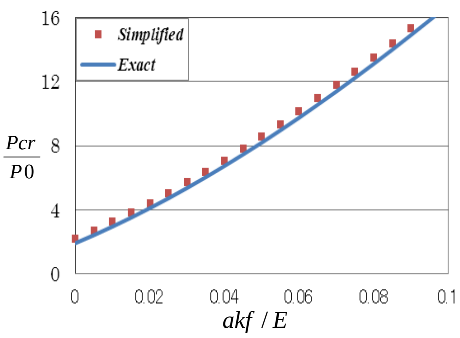

Fig.2 Comparison of p_{cr}

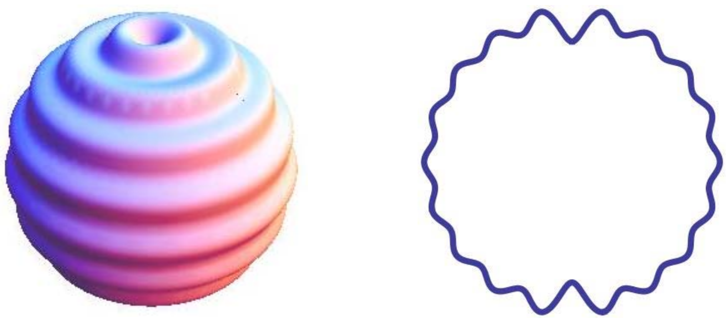

Fig.3 Buckling modes with $a/h=50$ and $ak_f/E=0$

5. SUMMARY

The characteristic buckling eigenmodes in hydrostatically pressurized complete spherical shells filled with an elastic medium have been demonstrated. A theoretical formulation based on small-displacement thin shell theory produced governing equations of equilibrium that has been solved using an exact methodology without any assumptions on axisymmetry or inextensibility. In addition, simplified formulations for estimating the critical pressure and the mode number have been proposed with a Rayleigh-Ritz approach. The expressions obtained from the Rayleigh-Ritz methodology have been shown to give sufficiently accurate results compared with the exact values.

ACKNOWLEDGEMENTS

This work was partially supported by KAKENHI (Grant-in-Aid for Young Scientists (A), No. 24686096) from MEXT, Japan.

REFERENCES

Yin, J., Cao, Z., Li, C., Sheinman, I. and Chen, X. (2008) "Stress-driven buckling patterns in spheroidal core/shell structures,"ProcNatlAcadSci USA, PNAS,**105**, 19132-5
Timoshenko, SP. and Gere, JM. (1961) "Theory of elastic stability," McGraw-Hill
Flugge, W. (1973)"Stresses in shells," Springer-Verlag
Fok, SL. and Allwright, DJ. (2001) "Buckling of a spherical shell embedded in an elastic medium loaded by a far-field hydrostatic pressure," J Strain Anal Eng Des, **36**, 535-44