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Three-dimensional free vibration analysis of functionally graded rectangular plates using the B-spline Ritz method

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ABSTRACT

This paper presents the three-dimensional free vibration analysis of functionally graded rectangular plates with arbitrary boundary conditions using the B-spline Ritz method based on the theory of elasticity. The Young's modulus of the plate is assumed to vary through the thickness direction, and the Poisson's ratio and density are constant. The convergence and accuracy of the present method were investigated. Rapid, stable convergence and high accuracy were obtained by the present method. Furthermore, the effects of thickness-to-length ratio, Young's modulus ratio and boundary condition on the fundamental frequency parameters, mode shapes, strain energies and kinetic energies of functionally graded plate were also carried out.

1. INTRODUCTION

Functionally graded materials (FGMs) were first proposed by a group of Japanese material scientists (Koizumi 1997). The material property of FGMs varies smoothly and continuously through the thickness from the surface to other surface. FGMs have received considerable attention in modern technology as a structural element or a material ingredient due to increasing performance demands, and are expected as a future high performance composite material in many engineering fields as aerospace, nuclear, marine, and civil engineering, etc. Therefore, an understanding of the dynamic behavior of the structure members made of FGMs is very important in structural design.

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The free vibration analysis of structural members based on the three-dimensional (3-D) theory of elasticity does not rely to hypotheses involving kinematics of deformation. Therefore, 3-D free vibration analysis provides realistic results based on the theory, and it can not be predicted by shear deformation theories (Mindlin 1951, Reddy 1985). The studies on the 3-D free vibrations of rectangular plates made of FGMs are limited to four edges simply-supported condition (Vel 2004). Recently, Malekzadeh (2009) proposed the semi-analytical differential quadrature method, and analyzed 3-D free vibrations of functionally graded rectangular plates with one pair of parallel simply-supported edges.

This paper presents the 3-D free vibration analysis of functionally graded rectangular plates with arbitrary boundary conditions using the B-spline Ritz method. The Young's modulus of the plate is assumed to vary through the thickness direction, and the Poisson's ratio and density are constant. The analysis is based on the linear, small-deformation 3-D theory of elasticity. The B-spline Ritz method has been proposed by Nagino (2008). This method is formulated by the Ritz procedure with triplicate series of normalized B-spline functions (Boor 1972) as displacement amplitude components. The geometric boundary conditions are numerically satisfied by the method of artificial spring (Kao 1974). To demonstrate the convergence and accuracy of the present method, several examples are solved, and the results are compared with the 3-D finite element solutions. Furthermore, a detailed investigation of the effects of thickness-to-length ratio, Young's modulus ratio and boundary condition on the fundamental frequency parameters, mode shapes, strain energies and kinetic energies of functionally graded plate are also carried out.

2. FORMULATION

2.1. Material properties of FGMs

The Young's modulus $E(z)$ in the functionally graded plate is assumed to vary exponentially in the thickness direction as follows:

$$E(z) = E_b \exp\left\{ p\left(\frac{z}{h}\right)\right\},\ \ p = \ln\left(\frac{E_t}{E_b}\right), \qquad (1)$$

where E_t and E_b are the Young's moduli at top and bottom surface, respectively. Poisson's ratio ν and density ρ are constant. The distribution of Young's modulus $E(z)$ in the thickness direction is shown in Fig. 1.

2.2. Analytical model

Consider a functionally graded rectangular plate with length a, width b and uniform thickness h in Fig. 2. The stress free surface are assumed at $z = 0$ and $z = h$. The plate is defined with respect to a right-handed orthogonal coordinate system (x, y, z). The periodic displacement components at any point are defined by the in-plane components u, v and the transverse component w in the x, y and z direction, respectively.

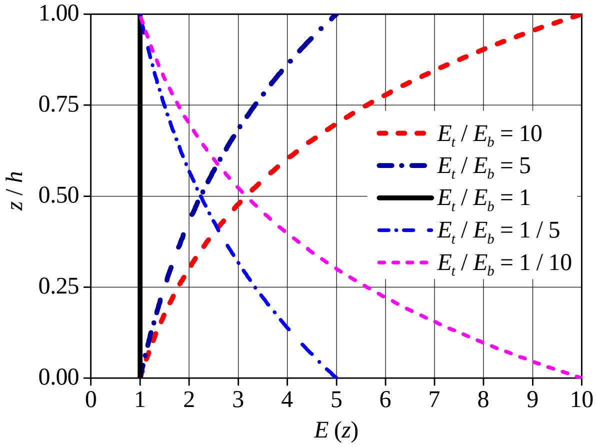


Fig. 1. The distribution of Young's modulus in the thickness direction

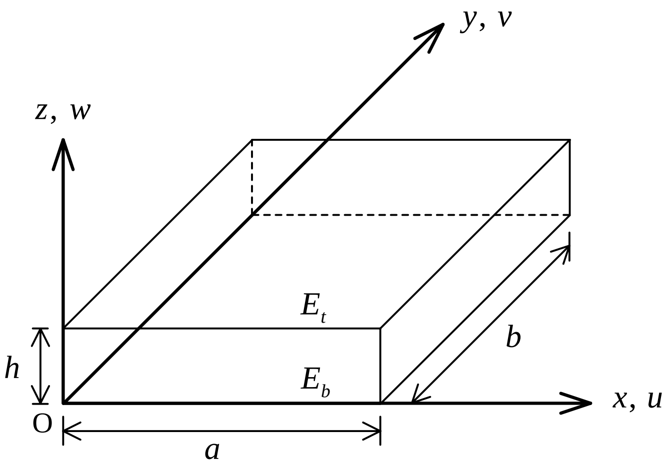


Fig. 2. Coordinate system and geometry of functionally graded plate

2.3. Formulation of eigenvalue equation based on the B-spline Ritz method

The strain energy $\bar{U}$ of the functionally graded rectangular plate can be expressed in integral form as

$$\bar{U} = \frac{1}{2}\int_0^a \int_0^b \int_0^h (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x\,, \tag{2}$$

in which ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, γ_{zx} and σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx} are strain and stress components, respectively.

In the 3-D theory of elasticity, strain and stress components of the functionally graded rectangular plate are defined as

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},\ \varepsilon_z = \frac{\partial w}{\partial z},\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\ \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \tag{3}$$

$$\sigma_x = \{\mu(z) + 2G(z)\}\varepsilon_x + \mu(z)\varepsilon_y + \mu(z)\varepsilon_z\,,$$
$$\sigma_y = \mu(z)\varepsilon_x + \{\mu(z) + 2G(z)\}\varepsilon_y + \mu(z)\varepsilon_z\,,$$
$$\sigma_z = \mu(z)\varepsilon_x + \mu(z)\varepsilon_y + \{\mu(z) + 2G(z)\}\varepsilon_z\,,$$
$$\tau_{xy} = G(z)\gamma_{xy}\,,\ \tau_{yz} = G(z)\gamma_{yz}\,,\ \tau_{zx} = G(z)\gamma_{zx}\,, \tag{4}$$

$$\mu(z) = \frac{\nu E(z)}{(1-2\nu)(1+\nu)}\,,\ G(z) = \frac{E(z)}{2(1+\nu)}\,, \tag{5}$$

where $E(z)$ is Young's modulus, ν is Poisson's ratio, and $G(z)$ is shear modulus.

The kinetic energy $\bar{T}$ of the plate can be written as

$$\bar{T} = \frac{\rho}{2}\int_0^a\int_0^b\int_0^h \left\{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right\} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,, \tag{6}$$

in which ρ is the mass density per unit volume.

Here, for simplicity and convenience in mathematical formulation, the following non-dimensional coordinate system (ξ, η, ζ) are introduced as

$$\xi = \frac{x}{a}\,,\ \eta = \frac{y}{b}\,,\ \zeta = \frac{z}{h}\,. \tag{7}$$

The periodic displacement components can be expressed by non-dimensional displacement amplitude function U, V and W in ξ, η and ζ coordinates and the temporal coordinate t as

$$u(\xi,\eta,\zeta,t) = aU(\xi,\eta,\zeta)e^{\mathrm{i}\omega t}\,,\ v(\xi,\eta,\zeta,t) = aV(\xi,\eta,\zeta)e^{\mathrm{i}\omega t}\,,\ w(\xi,\eta,\zeta,t) = aW(\xi,\eta,\zeta)e^{\mathrm{i}\omega t}\,, \tag{8}$$

where ω denotes the circular frequency and $\mathrm{i} = \sqrt{-1}$ is an imaginary constant.

The assumed spatial displacement field is based on a separable assumption for displacement amplitude functions. The functions are expressed as the summation of a triplicate series of B-spline functions as follows:

$$U(\xi,\eta,\zeta) = \sum_{m=1}^{i_\xi}\sum_{n=1}^{i_\eta}\sum_{r=1}^{i_\zeta} A_{mnr} N_{m,k_\xi}(\xi) N_{n,k_\eta}(\eta) N_{r,k_\zeta}(\zeta)\,,$$
$$V(\xi,\eta,\zeta) = \sum_{m=1}^{i_\xi}\sum_{n=1}^{i_\eta}\sum_{r=1}^{i_\zeta} B_{mnr} N_{m,k_\xi}(\xi) N_{n,k_\eta}(\eta) N_{r,k_\zeta}(\zeta)\,,$$
$$W(\xi,\eta,\zeta) = \sum_{m=1}^{i_\xi}\sum_{n=1}^{i_\eta}\sum_{r=1}^{i_\zeta} C_{mnr} N_{m,k_\xi}(\xi) N_{n,k_\eta}(\eta) N_{r,k_\zeta}(\zeta)\,, \tag{9}$$

in which $N_{m,k_\xi}(\xi)$, $N_{n,k_\eta}(\eta)$ and $N_{r,k_\zeta}(\zeta)$ are one-dimensional (1-D) normalized B-spline functions with the degree of spline function $(k_\xi - 1)$, $(k_\eta - 1)$ and $(k_\zeta - 1)$. A_{mnr}, B_{mnr} and C_{mnr} are unknown spline coefficients. The appearing in Eq. (9) are defined as: $i_{k_\xi} = m_\xi + k_\xi - 2$, $i_{k_\eta} = m_\eta + k_\eta - 2$ and $i_{k_\zeta} = m_\zeta + k_\zeta - 2$, where m_ξ, m_η, m_ζ and k_ξ, k_η, k_ζ are the number of knots and the order of spline function in the ξ, η and ζ directions, respectively.

Substituting Eqs. (7), (8) and (9) into Eqs. (2) and (6), the maximum strain energy $U_{\max}$ and maximum kinetic energy $T_{\max}$ of the plate can be written in a non-dimensional coordinate systems as

$$\begin{aligned} U_{\max} &= \frac{abhE_b}{2}\int_0^1\int_0^1\int_0^1 (\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})\,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &= \frac{abhE_b}{2}\{\Delta\}^{\mathrm{T}}[\mathrm{K_p}]\{\Delta\}, \end{aligned} \tag{10}$$

$$\begin{aligned} T_{\max} &= \frac{\rho\omega^2 a^3 bh}{2}\int_0^1\int_0^1\int_0^1 (U^2 + V^2 + W^2)\,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &= \frac{\rho\omega^2 a^3 bh}{2}\{\Delta\}^{\mathrm{T}}[\mathrm{M}]\{\Delta\}, \end{aligned} \tag{11}$$

where $[\mathrm{K_p}]$ and $[\mathrm{M}]$ are the stiffness and mass matrices of the plate, respectively, and $\{\Delta\}$ is unknown coefficient vector in the following:

$$\{\Delta\} = \left\{ \{\delta_A\} \quad \{\delta_B\} \quad \{\delta_C\} \right\}^{\mathrm{T}}, \tag{12}$$

in which the column vectors $\{\delta_A\}$, $\{\delta_B\}$ and $\{\delta_C\}$ as

$$\begin{aligned} \{\delta_A\} &= \{A_{111} \quad A_{112} \quad \cdots \quad A_{11i_\zeta} \quad A_{121} \quad \cdots \quad A_{12i_\zeta} \quad \cdots \quad A_{1i_\eta i_\zeta} \quad \cdots \quad A_{i_\xi i_\eta i_\zeta}\}^{\mathrm{T}}, \\ \{\delta_B\} &= \{B_{111} \quad B_{112} \quad \cdots \quad B_{11i_\zeta} \quad B_{121} \quad \cdots \quad B_{12i_\zeta} \quad \cdots \quad B_{1i_\eta i_\zeta} \quad \cdots \quad B_{i_\xi i_\eta i_\zeta}\}^{\mathrm{T}}, \\ \{\delta_C\} &= \{C_{111} \quad C_{112} \quad \cdots \quad C_{11i_\zeta} \quad C_{121} \quad \cdots \quad C_{12i_\zeta} \quad \cdots \quad C_{1i_\eta i_\zeta} \quad \cdots \quad C_{i_\xi i_\eta i_\zeta}\}^{\mathrm{T}}. \end{aligned} \tag{13}$$

The boundary conditions at the four edges (x = 0, a and y = 0, b) of a functionally graded rectangular plate would be satisfied as follows:

Simply supported:

$$\begin{aligned} &v = w = 0,\ \ \sigma_x = 0 \ \text{ at } \ (x = 0, a), \\ &u = w = 0,\ \ \sigma_y = 0 \ \text{ at } \ (y = 0, b). \end{aligned} \tag{14}$$

Clamped edge:

$$u = v = w = 0 \text{ at } (x = 0, a), \\ u = v = w = 0 \text{ at } (y = 0, b). \tag{15}$$

Free edge:

$$\sigma_x = \tau_{xy} = \tau_{xz} = 0 \text{ at } (x = 0, a), \\ \sigma_y = \tau_{yx} = \tau_{yz} = 0 \text{ at } (y = 0, b). \tag{16}$$

The boundary conditions for top and bottom stress free surfaces of the plate can be expressed by

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0 \text{ at } (z = 0, h). \tag{17}$$

To deal with the geometric boundary conditions at the four edges ($x = 0$, a and $y = 0$, b), the method of artificial spring (Kao 1974) is used. In this method, three types of spring coefficient α, β and γ corresponding to the geometric boundary conditions u, v, w are introduced at each boundary edges of the plate.

The energy contribution L due to the spring is given by

$$L = \frac{1}{2}\int_0^b\int_0^h (\alpha u^2 + \beta v^2 + \gamma w^2)\,\mathrm{d}z\,\mathrm{d}y\bigg|_{x=0,\,a} + \frac{1}{2}\int_0^a\int_0^h (\alpha u^2 + \beta v^2 + \gamma w^2)\,\mathrm{d}z\,\mathrm{d}y\bigg|_{y=0,\,b}. \tag{18}$$

Substituting Eqs. (7), (8) and (9) into Eq. (18), the maximum artificial spring energy $L_{\max}$ of the plate can be given in a non-dimensional coordinate systems as

$$L_{\max} = \frac{a^2bh}{2}\int_0^1\int_0^1 (\alpha U^2 + \beta V^2 + \gamma W^2)\,\mathrm{d}\zeta\,\mathrm{d}\eta\bigg|_{\xi=0,\,1} + \frac{a^3h}{2}\int_0^1\int_0^1 (\alpha U^2 + \beta V^2 + \gamma W^2)\,\mathrm{d}\zeta\,\mathrm{d}\xi\bigg|_{\eta=0,\,1} \\ = \frac{abhE_b}{2}\{\Delta\}^{\mathrm{T}}[\mathrm{K_L}]\{\Delta\}, \tag{19}$$

where $[\mathrm{K_L}]$ is the stiffness matrix for the artificial springs.

For the geometric boundary condition at the four edges ($\xi = 0, 1$ and $\eta = 0, 1$), the non-dimensional spring parameters are assumed to be zero, the boundary edges will become the stress free condition. If the spring parameter is assumed to be infinite, the boundary edges will lead to procedure the clamped condition.

The total potential energy Π of the functionally graded plate can be expressed as

$$\Pi = (U_{\max} + L_{\max}) - T_{\max}. \tag{20}$$

In Eq. (20), minimizing the total potential energy Π with respect to the unknown spline coefficient vector $\{\Delta\}$ i.e.:

$$\frac{\partial \Pi}{\partial \{\Delta\}} = 0, \quad (21)$$

which lead to the following the eigenvalue equation in matrix form:

$$\left(\begin{bmatrix} [\mathrm{K_{UU}}] & [\mathrm{K_{UV}}] & [\mathrm{K_{UW}}] \\ [\mathrm{K_{VU}}] & [\mathrm{K_{VV}}] & [\mathrm{K_{VW}}] \\ [\mathrm{K_{WU}}] & [\mathrm{K_{WV}}] & [\mathrm{K_{WW}}] \end{bmatrix} - \Omega^2 \begin{bmatrix} [\mathrm{M_{UU}}] & [0] & [0] \\ [0] & [\mathrm{M_{VV}}] & [0] \\ [0] & [0] & [\mathrm{M_{WW}}] \end{bmatrix} \right) \begin{Bmatrix} \{\delta_A\} \\ \{\delta_B\} \\ \{\delta_C\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix}, \quad (22)$$

$$\Omega^2 = \omega^2 a^2 \frac{\rho}{E_b}, \quad (23)$$

where Ω is the non-dimensional frequency parameter, $[\mathrm{K_{IJ}}]$ and $[\mathrm{M_{IJ}}]$ (I, J = U, V, W) are the sub-stiffness matrices and sub-mass matrices of the plate, respectively. The size of matrix in Eq. (22) is $3(m_\xi + k_\xi - 2)(m_\eta + k_\eta - 2)(m_\zeta + k_\zeta - 2)$.

3. NUMERICAL RESULTS

The fundamental natural frequencies of the functionally graded rectangular plates with arbitrary boundary conditions were analyzed. Firstly, the convergence and accuracy of the present method were investigated. Next, the effects of thickness-to-length ratio h / a, Young's modulus ratio E_t / E_b and boundary condition on the fundamental frequency parameters, mode shapes, strain energies and kinetic energies of functionally graded plate were also carried out. Poisson's ration $\nu = 0.3$ and aspect ratio $b / a = 1$ were used in the numerical calculations. For the definition of the boundary conditions of the plate, for example, the symbols SF-CS, identifies a plate with edges (ξ = 0, 1) and (η = 0, 1) having simply supported edge (S), free edge (F), clamped edge (C) and simply supported edge (S), respectively. The non-dimensional artificial spring parameters are used to be 10^8 and the placement of the knots was set to the Chebyshev-Gauss-Lobatto points, namely, the shifted Chebyshev points (Nagino 2008) in the following analysis.

All computations are performed in double precision on a personal computer, and all of the fundamental frequency parameter $\Omega_{1\mathrm{st}} = \omega_{1\mathrm{st}} a \sqrt{\rho / E_b}$ and vibration modes are organized to four significant digits.

3.1. Convergence and comparison studies

The Ritz method provides theoretically accurate solutions, and, generally, the natural frequencies obtained by the Ritz procedure are the upper bounds of the exact frequencies. The effects of the degree of spline function $(k_\xi - 1) \times (k_\eta - 1) \times (k_\zeta - 1)$ and the number of knot $m_\xi \times m_\eta \times m_\zeta$ on the convergence of the present method are investigated. Moreover, the accuracy of the present method is also investigated.

Table 1. Convergence of fundamental frequency parameter Ω_{1st} for SS-SS functionally graded plates: $E_t / E_b = 100$

h/a	$(k_\xi - 1)\times(k_\eta - 1)\times(k_\zeta - 1)$	$m_\xi \times m_\eta \times m_\zeta$	D.O.F.	Ω_{1st} (Error %)
0.2	3×3×2	15×15×13	12138	**3.352**
		17×17×13	15162	**3.352**
		19×19×13	18522	**3.352**
	4×4×3	15×15×13	14580	**3.352**
		17×17×13	18000	**3.352**
		19×19×13	21780	**3.352**
	3-D FEM (C3D8)	51×51×11	85833	3.331 (– 0.63 %)
		81×81×17	334611	3.343 (– 0.27 %)
0.5	3×3×2	15×15×13	12138	**6.334**
		17×17×13	15162	**6.334**
		19×19×13	18522	**6.334**
	4×4×3	15×15×13	14580	**6.334**
		17×17×13	18000	**6.334**
		19×19×13	21780	**6.334**
	3-D FEM (C3D8)	41×41×21	105903	6.328 (– 0.09 %)
		61×61×31	346053	6.332 (– 0.03 %)

Table 2. Convergence of fundamental frequency parameter Ω_{1st} for CC-CC functionally graded plates: $E_t / E_b = 100$

h/a	$(k_\xi - 1)\times(k_\eta - 1)\times(k_\zeta - 1)$	$m_\xi \times m_\eta \times m_\zeta$	D.O.F.	Ω_{1st} (Error %)
0.2	3×3×2	15×15×13	12138	**5.477**
		17×17×13	15162	**5.477**
		19×19×13	18522	**5.477**
	4×4×3	15×15×13	14580	5.477
		17×17×13	18000	**5.476**
		19×19×13	21780	**5.476**
	3-D FEM (C3D8)	51×51×11	85833	5.454 (– 0.40 %)
		81×81×17	334611	5.468 (– 0.15 %)
0.5	3×3×2	15×15×13	12138	**8.546**
		17×17×13	15162	**8.546**
		19×19×13	18522	**8.546**
	4×4×3	15×15×13	14580	8.546
		17×17×13	18000	**8.545**
		19×19×13	21780	**8.545**
	3-D FEM (C3D8)	41×41×21	105903	8.551 (+ 0.07 %)
		61×61×31	346053	8.548 (+ 0.04 %)

Table 3. Convergence of fundamental frequency parameter Ω_{1st} for CF-FF functionally graded plates: $E_t / E_b = 100$

h / a	$(k_\xi - 1)\times(k_\eta - 1)\times(k_\zeta - 1)$	$m_\xi \times m_\eta \times m_\zeta$	D.O.F.	Ω_{1st} (Error %)
0.2	3×3×2	15×15×13	12138	0.6294
		17×17×13	15162	**0.6293**
		19×19×13	18522	**0.6293**
	4×4×3	15×15×13	14580	**0.6293**
		17×17×13	18000	**0.6293**
		19×19×13	21780	**0.6293**
	3-D FEM (C3D8)	51×51×11	85833	0.6280 (– 0.21 %)
		81×81×17	334611	0.6289 (– 0.06 %)
0.5	3×3×2	15×15×13	12138	**1.386**
		17×17×13	15162	**1.386**
		19×19×13	18522	**1.386**
	4×4×3	15×15×13	14580	**1.386**
		17×17×13	18000	**1.386**
		19×19×13	21780	**1.386**
	3-D FEM (C3D8)	41×41×21	105903	1.387 (+ 0.07 %)
		61×61×31	346053	1.387 (+ 0.07 %)

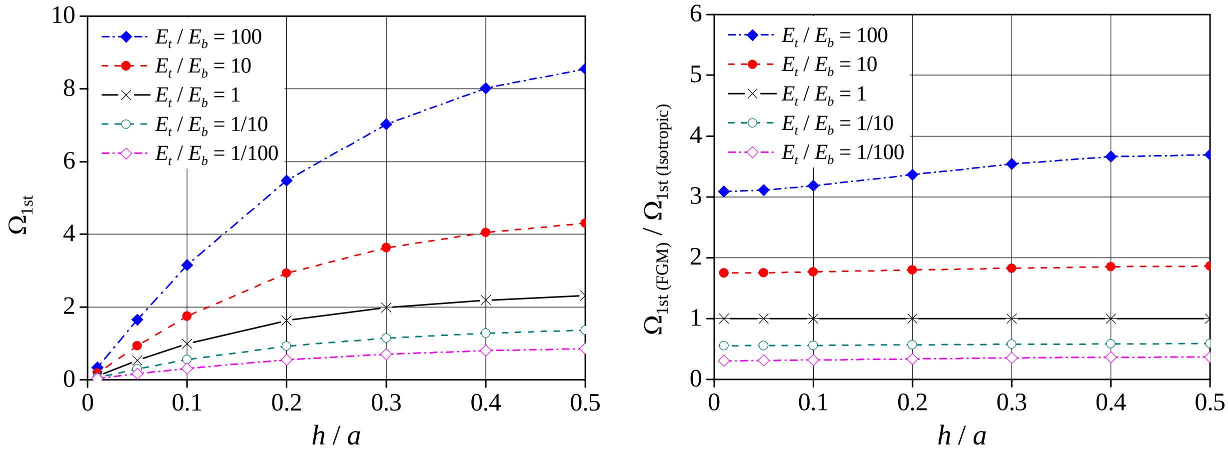
Tables 1, 2 and 3 show the effects of the degree of spline function $(k_\xi - 1) \times (k_\eta - 1) \times (k_\zeta - 1)$ and the number of knot $m_\xi \times m_\eta \times m_\zeta$ on the convergence of the fundamental frequency parameter Ω_{1st} for functionally graded square plate having SS-SS, CC-CC and CF-FF respectively. The thickness-to-length ratio h / a are set as 0.2 (moderately thick plate) and 0.5 (thick plate), and the Young's modulus ratio $E_t / E_b = 100$ is used. The results are compared with the 3-D finite element solutions by Abaqus 6.12. Here, C3D8 means first-order solid element. The error (%) is defined as follows:

$$\text{Error (\%)} = \frac{\Omega_{\text{FEM}} - \Omega_{\text{Present}}}{\Omega_{\text{Present}}} \times 100\,, \tag{24}$$

where Ω_{Present} is convergence value obtained from present method in Tables 1, 2 and 3.

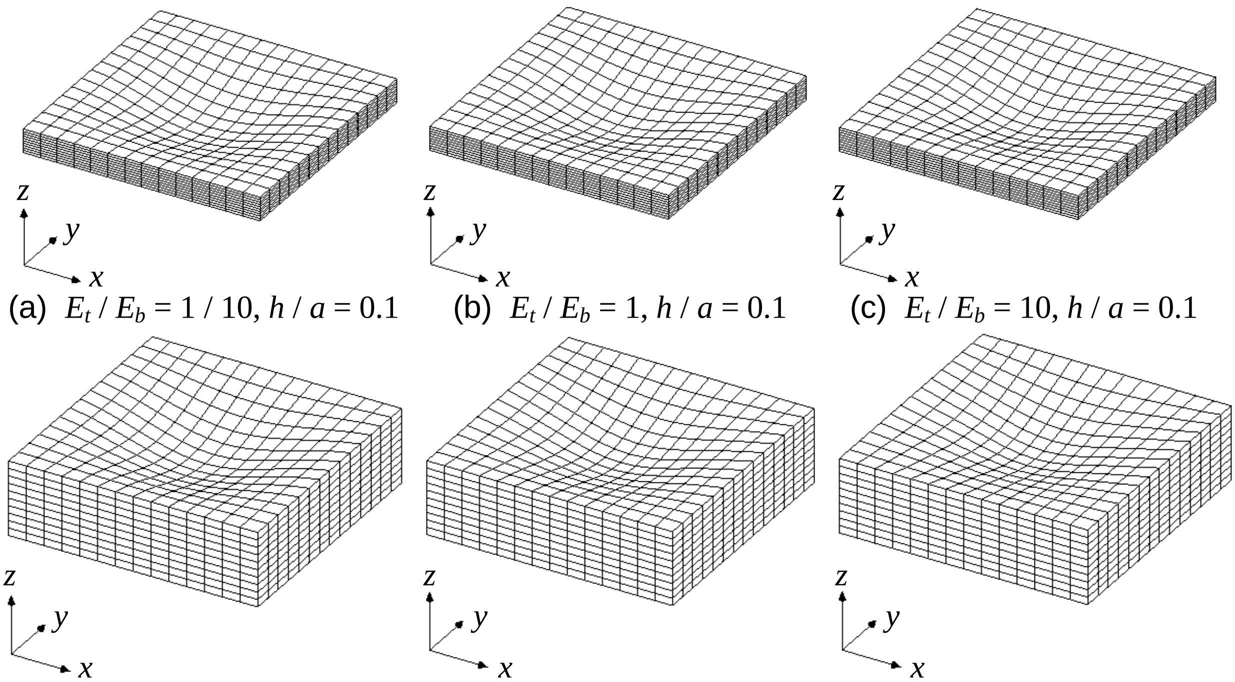
Tables 1, 2 and 3 show that stable convergence can be obtained. It is found that the fundamental frequency parameter Ω_{1st} rapidly converge by using the degree of spline function $4\times4\times3$. The results in Tables 1, 2 and 3 show excellent agreements in all cases; thus, high accuracy is obtained.

The B-spline Ritz method yields highly accurate results with few degrees of freedom (D.O.F.) for the fundamental frequencies and displacement mode shapes of the functionally graded rectangular plate. In addition, stable and rapidly converging and excellent upper bound solutions are obtained by the present method regardless of the thickness-to-length ratio and boundary conditions.



(a) Ω_{1st} (b) $\Omega_{1st\,(FGM)} / \Omega_{1st\,(Isotropic)}$

Fig. 3. The effects of thickness-to-length ratio and Young's modulus ratio on the fundamental frequency parameter of CC-CC functionally graded square plates.



(a) $E_t / E_b = 1 / 10$, $h / a = 0.1$ (b) $E_t / E_b = 1$, $h / a = 0.1$ (c) $E_t / E_b = 10$, $h / a = 0.1$

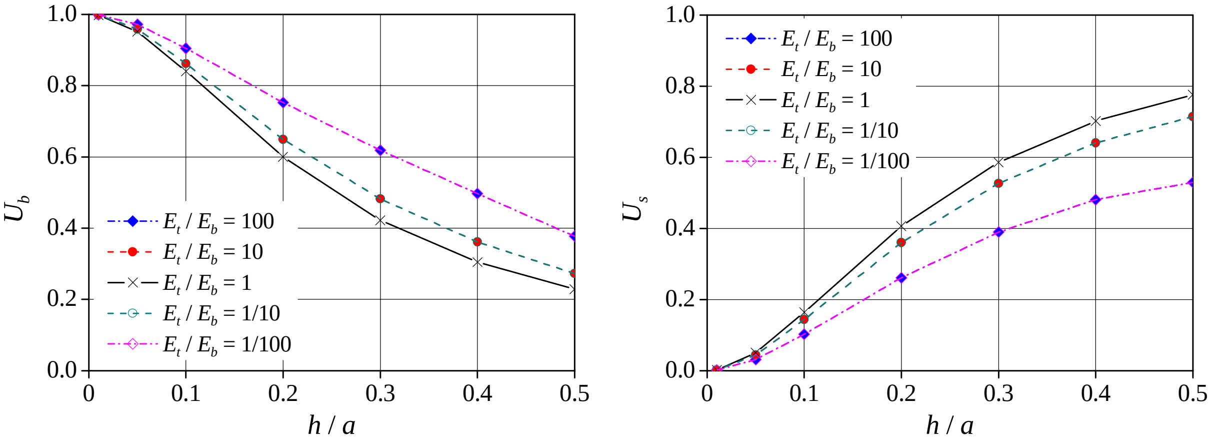
(d) $E_t / E_b = 1 / 10$, $h / a = 0.3$ (e) $E_t / E_b = 1$, $h / a = 0.3$ (f) $E_t / E_b = 10$, $h / a = 0.3$

Fig. 4. The fundamental modes of CC-CC functionally graded square plates.

3.2. Results and discussions

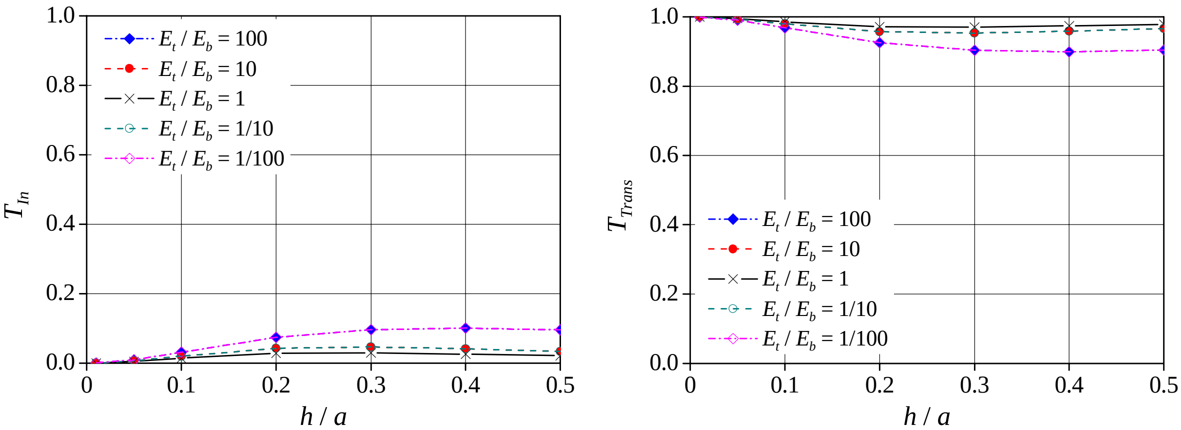
The B-spline Ritz method is applied to investigate the fundamental free vibration of the functionally graded square plate with arbitrary boundary conditions.

Fig. 3 shows the effects of thickness-to-length ratio h / a and Young's modulus ratio E_t / E_b on the fundamental frequency parameter Ω_{1st} of CC-CC functionally graded



(a) Bending component (b) Transverse shear component

Fig. 5. The effects of thickness-to-length ratio and Young's modulus ratio on the strain energies of fundamental mode for CC-CC functionally graded square plates.



(a) In-plane component (b) Transverse-plane component

Fig. 6. The effects of thickness-to-length ratio and Young's modulus ratio on the kinetic energies of fundamental mode for CC-CC functionally graded square plates.

square plates. In addition, the fundamental modes of CC-CC functionally graded square plates are depicted in Fig.4. The thickness-to-length ratio h / a varies from 0.01 to 0.5. The Young's modulus ratio E_t / E_b is set as 100, 10, 1, 1/10, and 1/100. Note that $E_t / E_b = 1$ means an isotropic plate.

It is found that the fundamental frequencies parameter Ω_{1st} increases with increment of thickness-to-length ratio h / a and Young's modulus ratio E_t / E_b.

Fig. 5 shows the effects of thickness-to-length ratio h / a and Young's modulus ratio E_t / E_b on the strain energies of fundamental vibration mode for CC-CC functionally graded square plates. The numerical calculation condition is the same as the Fig. 3. Here, the bending component U_b and the transverse shear component U_s are defined by

$$U_b = \frac{1}{2}\int_0^a\int_0^b\int_0^h (\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy})\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,, \tag{25}$$

$$U_s = \frac{1}{2}\int_0^a\int_0^b\int_0^h (\tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,. \tag{26}$$

It is seen that the bending component U_b decreases with increment of thickness-to-length ratio h / a, however, the bending component U_b of functionally graded plates are greater than the ones of isotropic plates. On the other hand, with increasing thickness-to-length ratio h / a, the transverse shear component U_s also increases. But, the transverse shear component U_s of functionally graded plates are smaller than the ones of isotropic plates. Thus, it seems that functionally graded materials are strong to transverse shear deformation.

Fig. 6 depicts the effects of thickness-to-length ratio h / a and Young's modulus ratio E_t / E_b on the kinetic energies of fundamental mode for CC-CC functionally graded square plates. The numerical calculation condition is the same as the Fig. 3. Here, the In-plane component T_{In} and the transverse-plane component T_{Trans} are defined by

$$T_{In} = \frac{\rho\omega^2}{2}\int_0^a\int_0^b\int_0^h (u^2 + v^2)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,, \tag{27}$$

$$T_{Trans} = \frac{\rho\omega^2}{2}\int_0^a\int_0^b\int_0^h w^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,. \tag{28}$$

It is seen that the In-plane component T_{In} increases with increment of thickness-to-length ratio h / a and Young's modulus ratio E_t / E_b. On the other hand, the transverse-plane component T_{Trans} decreases with increment of thickness-to-length ratio h / a and Young's modulus ratio E_t / E_b.

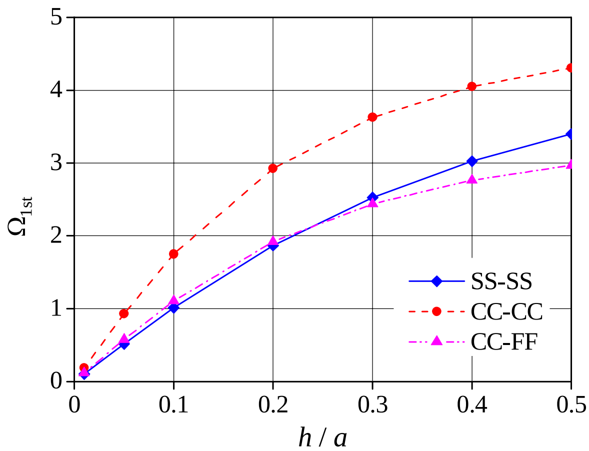


Fig. 7. The effects of thickness-to-length ratio and boundary condition the fundamental frequency parameter of functionally graded square plates.

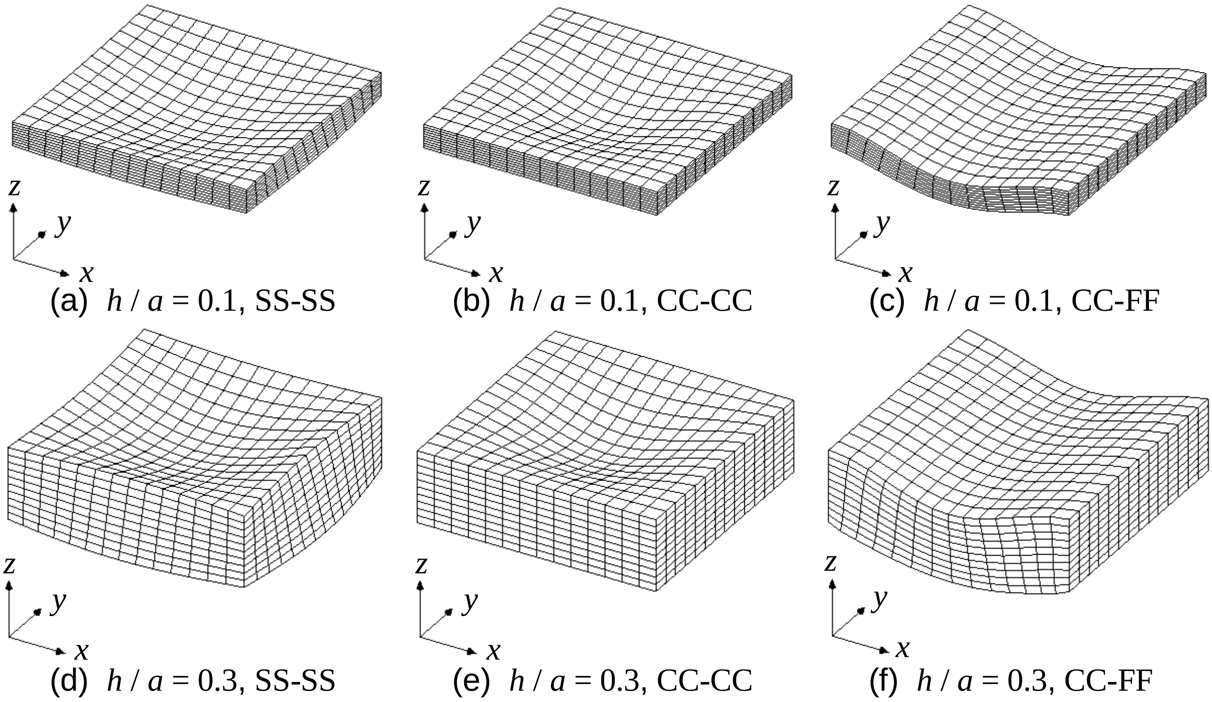


Fig. 8. The fundamental mode shapes of functionally graded thick plates with SS-SS, CC-CC and CC-FF.

Fig. 7 shows the effects of thickness-to-length ratio h / a and boundary condition the fundamental frequency parameter Ω_{1st} of functionally graded square plates. The thickness-to-length ratio h / a varies from 0.01 to 0.5. The Young's modulus ratio E_t / E_b is set as 10. In addition, Fig.8 depicts the fundamental mode shapes of functionally graded square plates having SS-SS, CC-CC, CC-FF.

From Fig. 7, with increasing the thickness-to-length ratio h / a, the fundamental frequencies parameter Ω_{1st} also increases regardless of the boundary conditions.

4. COCLUSIONS

The 3-D free vibration analysis of functionally graded rectangular plates with arbitrary boundary conditions has been presented. The governing eigenvalue equation is formulated by the B-spline Ritz method based on the linear, small-deformation 3-D theory of elasticity. Rapid, stable convergence and high accuracy were obtained by the present method. Furthermore, the effects of thickness-to-length ratio, Young's modulus ratio and boundary condition on the fundamental frequency parameters, mode shapes, strain energies and kinetic energies of functionally graded plate were also investigated. The fundamental frequencies parameter increases with increment of thickness-to-length ratio and Young's modulus ratio. Therefore, when the top surface sets higher Young's modulus, the functionally graded plates can increase the natural frequency than the isotropic plates. Moreover, functionally graded materials are strong to transverse shear deformation. Finally, with increasing the thickness-to-length ratio, the fundamental frequencies parameter also increases regardless of the boundary conditions.

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