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Similitude theory and criteria of long-span bridge aeroelasticity in wind tunnel testing

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ABSTRACT

Similitude theory plays an important role in the scaled bridge structure wind tunnel tests. The similitude between the prototype and the scaled model is achieved by keeping the dimensionless similitude criteria of these two systems similar. The methods commonly used to derive these similitude criteria could be categorized as dimensional analysis method, ratio of forces method and equation analysis method. This paper reviewed these three traditional methods and revealed their respective mechanisms of realizing similitude. A new deriving method, named Lie Group method, was proposed based on the Lie Group theory. By introducing Lie Groups and other related mathematical concepts, Lie Group method is able to determine kinds of symmetric mechanisms of Navier-Stokes equations. The scaled operations in bridge structure wind tunnel tests correspond to the second scaling symmetry group. Determination of invariants under the second scaling symmetry group was followed. It was found that the invariants are exactly the common similitude criteria. Considering the implicit connection with the fluid system in the flutter force model, a compatibility check of symmetric mechanisms between a structural systemand the fluid system around it was carried out. And the invariants of the structural system were solved.

1. INTRODUCTION

With the ever-growing of span length, bridge structures are becoming lighter, more flexible and in particular, more sensitive to wind actions. Structural vibration related to wind actions is thus a major concern of long span bridges. In terms of the driving mechanisms of wind induced vibration, there are two kinds, namely aerodynamic and aeroelastic vibrations(Scanlan 1978; Scanlan 1978). Both kinds of vibrations involve complex interactions between wind and bridge structure withdifferent interactions. Currently, it is still impossible to adopt a completely analytical approach to study this physical phenomenon. Only semi-analytical methods are available which essentially present empirical aspects. Although this semi-analytical way is well developed for past years, it is still unable to provide reasonable explanations to some wind induced

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phenomena, especially for aeroelastic vibration(Kareem *et al.* 2013). With the rapid development of computer science, computational fluid dynamics (CFD) provides a new way to study the problem of bridge and wind interaction(Larsen *et al.* 1998; Panneer *et al.* 1998). However, considering the complex geometries of bridge girders attributed to the accessory components attached on the girder and the elasticity inducing deformation of bridge structures, the requirements of enormous computational resources and time make a completely numerical simulation of this interaction prohibitive.

Because of the dilemmas faced by the above two approaches, physical experiments conducted in wind tunnel have played an important and sometimes even dominant role in the evaluation of wind induced vibration. Wind tunnel tests not only directly reveal the wind sensibility of bridge structures, but also provide required information to the theoretical analysis. In wind tunnel tests, scaled bridge structure models are employed to simulate the interactions taking place in the corresponding prototypes(Miyata *et al.* 1992; Diana *et al.* 1995). To correctly simulate interactions, it is important to make sure the similarity of the flow patterns between a bridge structure and the surrounding flow. In general, similitude theory is the guarantee to make the similitude requirements be satisfied, and also provides a law on predicting the prototype performance based on the scaled model observations. The importance of similitude theory in wind tunnel testing cannot be overemphasized.

Completely founded around 1930s, the core of similitude theory consists of three similitude theorems, including the basic properties of two similitude phenomena, a derivation approach of similitude criteria, and the sufficiency condition of similitude phenomena. Among them, the second theorems, sometimes referred as Buckingham Π Theorem, is most widely adopted since it provides a practical introduction on conducting scaled experiments which satisfy similitude requirements(Szücs 1980).

According to the Π Theorem, three kinds of method for deriving similitude criteria of wind tunnel tests have already been proposed. These three methods are the dimensional analysis method, the ratio of forces method and the equation analysis method, respectively. All three methods originate from different aspects, but obtain the same similitude criteria when they are applied to determine the similitude conditions under which the scaled model tests are carried out.

For a scaled bridge structure test, three similitude requirements, namely geometric similarity, kinematic similarity and dynamic similarity, theoretically need to be obeyed under the assumption of incompressible Newtonian fluid with constant temperature. Unfortunately, the latter two requirements are inevitably violated in wind tunnel tests due to the well-known scaled effects (Larose *et al.* 2006; Larsen *et al.* 2008). Therefore, wind tunnel tests of scaled bridge structures are impossible to achieve strict similitude and essentially partial similitude. In terms of the derivation of similitude criteria, the above three named traditional methods all provide simple procedures. However, when the essence of partial similitude is considered, their incompetence of strictly revealing the mechanism of partial similitude will be shown. Besides, the lack of connection with the theoretical frameworks of bridge aerodynamic and aeroelastic analysis is another obvious defect. Since semi-empirical force models which only involve the structural information are adopted in the analytical method, it seems that the bridge structures and the surrounding fluid are two separate systems which have no connection between each other, especially for bridge flutter divergence and vortex induced vibration. It is thus necessity to check the similitude criteria compatibility between a structure and the fluid around it. A deriving method applicable to both fluid field and a bridge structure is favorable to fulfill this check. Therefore, the main objective of this paper is to propose a new similitude criteria deriving approach which is able to overcome the above two mentioned defects of the traditional deriving methods.

2. SIMILITUDE CRITERIA OF WIND TUNNEL TESTES

In order to predict bridge prototype performance under wind action as reliable as possible, it is necessary to accurately reproduce the prototype flow patterns in the scaled bridge structure wind tunnel tests. According to the fluid mechanics, a total of six variables are required to describe a fluid field, includingpressure, density, temperature, and three components of velocities. There are corresponding six basic equations to determine the set of six variables. And these six fundamental equations are the continuity equation, the Navier-Stokes (NS) equations in three directions, the energy conservation equation and the thermo-dynamical equation of state, respectively.

In bridge aerodynamics, air is assumed to be an incompressible Newtonian fluid with constant temperature. This assumption helps reduce the basic governing equations and the variables both from six to four. Only the continuity equation andthree NS equations, and the pressure and the corresponding fluid velocities are taken into account. Since the continuity equation is able integrated into the NS equations, only the NS equations are focused when equation analysis method is employed to derive the similitude criteria of scaled bridge structure wind tunnel tests.

Thanks to the efforts made by the previous experts, there are four important dimensionless similitude criteria for scaled bridge structure wind tunnel tests. In the scaled model tests, to fulfill the similitude requirements, keeping all these four dimensionless quantities equal for both the scaled modal and the prototype is the demanded practice. And these four dimensionless similitude criteria are Strouhal number St , Euler number Eu, Froude number Fr, and Reynolds number Re, respectively.

3. TRADITIONAL DERIVING METHODS

In general, there are three kinds of traditional similitude criteria deriving methods, including the Dimensional Analysis Method (DAM), the Ratio of ForcesMethod(RFM) and the Equation Analysis Method, which are briefly introduced afterwards.

3.1 Dimensional analysis method

Dimensional analysis method is widely adopted in engineering and science. According to the facts that the dimensions on both left and right sides of physical equations should be equal, this method presents the related physical quantities with fundamental dimensions(Szücs 1980). When it is applied to wind tunnel tests, the improved form, Buckingham ΠTheorem, derived from the general dimensional analysis method, is more preferred(Brand 1957). TheΠ theorem states that the number of dimensionless quantities π_i for a physical phenomenon is equal to the difference between the amount of related quantities and the rank of the dimensional matrix whose entries are the exponents of fundamental dimensions that are used to represent the related physical quantities.

For a scaled bridge model immersed in wind flow, the basically related physical quantities are flow density ρ, flow speed U, characteristic length D, frequency n, gravitational acceleration g, pressure p, and dynamic viscosity μ. Other related physical variables are functions of these seven quantities and could be presented as the form of

$$\mathcal{F}(\rho, U, D, n, \mu, g, p) = 0 \tag{1}$$

This equation has three fundamental dimensions, namely length l, time t and mass m. Thus the rank for the dimension matrix is 3. According to theΠ theorem, a total number of 7-3=4 dimensionless quantities could be derived.And Eq.1 can be re-expressed as

$$\mathcal{F}(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \tag{2}$$

where $\pi_1 = nD/U$, $\pi_2 = \mu/(\rho UD)$, $\pi_3 = gD/U^2$ and $\pi_4 = p/(\rho U^2)$. Observing the formulae of these four dimensionless quantities $\pi_i(i = 1, 2, 3, 4)$, it is easy to know that they are the exact dimensionless criteria, St , Re, $Fr$and Eu.

Obviously, theΠ theorem is a convenient method for deriving similitude criteria. It's even applicable to the situations where physical equations are unknown. However, dimensionless quantities derived by this method are not always unique and meaningful. Furthermore, when dimensionless quantities, such as damping ratio, aeroelastic parameters and so on, appear in the physical phenomenon, the derived results will become confusing because of the possibly arbitrary combinations between these dimensionless quantities.

3.2 Ratio of forces method

If the physical laws involved in a phenomenon are already known, the ratio of forces method could be used to derive the dimensionless similitude criteria. By constructing ratios between different kinds of forces that are formulated with the basically related physical quantities according to the physical laws, the physically meaningful dimensionless criteria are obtained(Massey 2006).

The example of scaled bridge model immersed in wind flow is adopted again, and the seven related physical quantities are the same as above mentioned. The physical laws involved in this phenomenon consist of Newton's second law of motion, Newton's law of viscosity and pressure formula. Selecting the convective inertial force as the reference quantity, four ratios could be calculated.

$$\pi_1 = \frac{F_{li}}{F_{ci}} = St, \qquad \pi_2 = \frac{F_{ci}}{F_v} = Re, \qquad \pi_3 = \frac{F_{ci}}{F_g} = Fr^2, \qquad \pi_4 = \frac{F_p}{F_{ci}} = Eu, \tag{3}$$

where F_{ci}, F_{υ}, F_g, F_p and F_{li} are the convective inertial force, the viscosity force, the gravity force, the pressure and the local inertial force, respectively.

It is clear that the dimensionless criteria are represented as ratios of related forces in the law analysis method. The similitude requirements will be achieved when the above four ratios are kept the same for both the prototype and the scaled model. Analyzing the essence of this method, these ratios are an alternative form of dynamic similitude. The similitude criteria derived by this method are physically meaningful, but the same problems for arbitrary force combinations which may result in confusing dimensionless quantities still exist.

3.3 Equation analysis method

Equation analysis method is applicable to the situation where the basic governing equations of the physical phenomenon are obtained. By introducing some reference variables, the original governing equations are transformed to their dimensionless counterparts. The coefficients in the dimensionless equations are defined as similitude criteria in this method(Tanaka 2003).

For the above example, the basic equations consist of the NS equations for the fluid and the equation of motion for the scaled model structure. For the reasons that only the dimensions of the physical quantities are considered in the dimensionless operation and NS equations contain all types of dimensions, it's reasonable to only focus on the NS equations when the equation analysis method is used to determine the similitude criteria. With incompressible flow and constant viscosity, the NS equations are as follows

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = \boldsymbol{f} - \frac{1}{\rho}\nabla \boldsymbol{p} + \upsilon \nabla^2 \boldsymbol{V} \tag{4}$$

To transform Eq.4 to be dimensionless, four reference physical quantities are introduced, namely the reference length L, the reference speed U, the reference press P and the reference frequency N. Substitute these references variables into Eq.4 and the corresponding dimensionless ones are of the forms

$$St\frac{\partial \boldsymbol{V_d}}{\partial t_d} + (\boldsymbol{V_d} \cdot \nabla_{\mathrm{d}})\boldsymbol{V_d} = \frac{\boldsymbol{f_d}}{Fr} - Eu \cdot \nabla_{\mathrm{d}} \boldsymbol{p} + \frac{\nabla_d^2 \boldsymbol{V_d}}{Re} \tag{5}$$

where the subscript 'd' represents a dimensionless correspondence of the original physical quantity. It is obvious that the coefficients in Eq.5 are the similitude criteria.

Based on the basic governing equations, the similitude criteria derived by this method presents clearly physical meaning. When these four coefficients are kept unaltered for the scaled model and the prototype, the governing equations of these two systems are the same. Concretely speaking, the equation analysis method provides a mechanism under which the scaled model and the prototype are described by the same governing equations, and make the dimensionless similitude criteria become connections between the two systems. This is a vivid explanation to the essence of similitude requirements provided by the equation analysis method. However, the success of this method mainly depends on the appropriate introduction of reference quantities which are purposely selected with a certain level of arbitrariness. Additionally, incompetence of properly dealing with dimensionless parameters also needs to be noticed.

4 LIE GROUPS METHOD

As a modern mathematical concept, Lie Groups play an indispensable role in analyzing the continuous symmetries of mathematical objects and structures, especially for the differential equations(Sagle *et al.* 1973; Olver 2000). Being a set of complicated partial differential equations based on important physical phenomena, the NS equations are widely studied under the powerful framework of Lie Groups(Fushchych *et al.* 1994; Fushchych *et al.* 1994; Razafindralandy *et al.* 2007; Razafindralandy *et al.* 2007). The Lie Groups that describe the symmetry of NS equations are also called symmetry groups. In terms of symmetry of the NS equations, it could be briefly explained as follows(Olver 2000; Razafindralandy, Hamdouni *et al.* 2007).

4.1 Symmetry of NS Equations

Set $\boldsymbol{y} = (t, \boldsymbol{x}, \boldsymbol{u}, p)$ as a set of the physical quantities in the NS equations, and re-express the NS equations as Eq.6 for simplicity.

$$\mathcal{F}(\boldsymbol{y}) = 0 \tag{6}$$

The one-parameter transformation T_ϵ defined in Eq. 7 is called as a symmetry of Eq.6 when the condition listed in Eq.8 is satisfied.

$$T_\epsilon : \boldsymbol{y} \mapsto \widehat{\boldsymbol{y}} = \widehat{\boldsymbol{y}}(\boldsymbol{y}, \epsilon) \tag{7}$$

in which ϵ is a parameter.

$$\mathcal{F}(\boldsymbol{y}) = 0 \;\Leftrightarrow\; \mathcal{F}(\widehat{\boldsymbol{y}}) = 0 \tag{8}$$

It means that the symmetry obtains a given solution of Eq.6 to another solution. The set of all the symmetries of Eq.6 constitutes a one-parameter symmetry group.Under the symmetry group actions, the variables in Eq.6 are transformed from the original set $\boldsymbol{y}$ to the new set $\widehat{\boldsymbol{y}}$. While there are some quantities related with $\boldsymbol{y}$, they keep unaltered under the transformation T_a. These quantities are defined as *invariants* corresponding to T_a.

Every symmetry group associates with certain vector fields. The one represents the variation of $\mathrm{y}$under the transformation $\mathrm{T_a}$ around $\mathrm{a} = 0$ is called infinitesimal generator and has a form of

$$X = \left.\frac{\partial \widehat{\boldsymbol{y}}}{\partial x}\right|_{a=0} = \sum_i \xi_i \frac{\partial}{\partial y_i} \tag{9}$$

where $\xi_i = \partial y_i/\partial a|_{a=0}$.

Finally, an equivalent form of Eq.6 could be derived according to Lie's fundamental theorem.

$$\mathcal{F}(\boldsymbol{y}) = 0 \iff X\mathcal{F}(\boldsymbol{y}) = 0 \tag{10}$$

4.2 Infinitesimal generators of NS Equations

The traditional treatment of combining body forces with pressure forces makes body forces implicit in the NS equations. The previous researches on symmetry of the NS equations were subsequently lack of the information on body forces transformation induced by a given symmetry group. For the specific case that a symmetry analysis to the coupling system of fluid and gravitationally dependent structures is carried out, the body force, in a precise sense, the gravity force results in a significant effect. The symmetry groups without explicit transformations of body force are difficult used in this case. The fact that a vast majority of bridge aeroelastic studies focus on the gravitationally dependent cable supported bridges makes an addition of body force transformation to the previously obtained symmetry groups more necessary. To achieve this goal, the body forces are re-expressed as gradients of potential energy ∇F. It's reasonable for this replacement when the conservative property of body forces is considered. After the replacement, the NS equations are of the form

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = -\nabla F - \frac{1}{\rho}\nabla \boldsymbol{p} + \upsilon \nabla^2 \boldsymbol{V} \tag{11}$$

Based on the above framework of Lie Groups theory, five categories, a total of nine, infinitesimal generators of Eq.11 could be obtained(Pukhnach 1972).

$$X_0 = \frac{\partial}{\partial t} \tag{12a}$$

$$Y_0 = \zeta(t)\left(\frac{\partial}{\partial p} + \frac{\partial}{\partial F}\right) \tag{12b}$$

$$X_{ij} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j} + u_j \frac{\partial}{\partial u_i} - u_i \frac{\partial}{\partial u_j}, \qquad (i = 1,2, j > i) \tag{12c}$$

$$X_i = \alpha_i(t)\frac{\partial}{\partial x_i} + \dot{\alpha}_i(t)\frac{\partial}{\partial u_i} - \rho x_i \ddot{\alpha}_i(t)\left(\frac{\partial}{\partial p}\right) - x_i \ddot{\alpha}_i(t)\frac{\partial}{\partial F}, \qquad (i = 1,2,3) \tag{12d}$$

$$S_1 = 2t\frac{\partial}{\partial t} + \sum_{j=1}^{3}\left(x_j \frac{\partial}{\partial x_j} - u_j \frac{\partial}{\partial u_j}\right) - 2p\frac{\partial}{\partial p} - 2F\frac{\partial}{\partial F} \tag{12d}$$

$$S_2 = \frac{\partial}{\partial t} + \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{3} u_i \frac{\partial}{\partial u_i} + 2F\frac{\partial}{\partial F} + 2p\frac{\partial}{\partial p} + 2\upsilon\frac{\partial}{\partial \upsilon} \tag{12e}$$

where ζ and $\boldsymbol{\alpha}$are arbitrary functions.

Correspondingly, there are five categories of symmetry groups which represent different symmetric mechanisms.

Time translation group:

$$G_{X_0}: (t, x, u, F, p) \mapsto (t + \epsilon, \boldsymbol{x}, \boldsymbol{u}, p, F) \tag{13a}$$

Pressure and conservative force translation group:

$$G_{Y_0}: (t, x, u, F, p) \mapsto \big(t, \boldsymbol{x}, \boldsymbol{u}, , F - \epsilon\zeta(t), p - \epsilon\zeta(t)\big) \tag{13b}$$

Rotation group:

$$G_{X_{ij}}: (t, x, u, F, p) \mapsto (t, R\boldsymbol{x}, R\boldsymbol{u}, F, p) \tag{13c}$$

Generalized Galilean group:

$$G_{X_i}: (t, x, u, F, p) \mapsto \begin{pmatrix} t, \boldsymbol{x} + \epsilon\boldsymbol{\alpha}(t), \boldsymbol{u} + \epsilon\dot{\boldsymbol{\alpha}}(t), \\ p - \epsilon\rho\boldsymbol{x} \cdot \ddot{\boldsymbol{\alpha}}(t) - \frac{1}{2}\epsilon^2\rho\boldsymbol{\alpha}(t) \cdot \ddot{\boldsymbol{\alpha}}(t) \\ F - \epsilon\boldsymbol{x} \cdot \ddot{\boldsymbol{\alpha}}(t) - \frac{1}{2}\epsilon^2\boldsymbol{\alpha}(t) \cdot \ddot{\boldsymbol{\alpha}}(t) \end{pmatrix} \tag{13d}$$

First scaling group:

$$G_{S_1}: (e^{2\epsilon}t, e^{\epsilon}\boldsymbol{x}, e^{-\epsilon}\boldsymbol{u}, e^{-2\epsilon}F, e^{-2\epsilon}p) \tag{13e}$$

Second scaling group:

$$G_{s2}: (t, \boldsymbol{x}, \boldsymbol{u}, F, p, \upsilon) \longmapsto (t, e^{\epsilon}\boldsymbol{x}, e^{\epsilon}\boldsymbol{u}, e^{2\epsilon}F, e^{2\epsilon}p, e^{2\epsilon}\upsilon) \tag{13f}$$

whereϵ is a parameter and R is a constant rotation matrix.

Because of the treatment to the body forces $\boldsymbol{f}$, these symmetry groups explicitly involves the transformation of body force. In other words, the symmetry groups are applicable to the cases containing the aeroelastic effects. Among those symmetry groups, the second scaling symmetry groups taking into account an action on the kinematic viscosity υ are the potential ones that are able to be associated with the scaled operations of wind tunnel tests.

4.3 Invariants based on Lie Group theory

Based on the infinitesimal generator S_2, the invariants under this scaling group action is able to be determined. According to the Lie Group theory, an invariant ζ of scaling group G_{s2} is a solution of the following linear, homogeneous first order partial differential equation,

$$S_2(\zeta) = \frac{\partial\zeta}{\partial t} + \sum_{i=1}^{3} x_i \frac{\partial\zeta}{\partial x_i} + \sum_{i=1}^{3} u_i \frac{\partial\zeta}{\partial u_i} + 2F\frac{\partial\zeta}{\partial F} + 2p\frac{\partial\zeta}{\partial p} + 2\upsilon\frac{\partial\zeta}{\partial\upsilon} = 0 \tag{14}$$

Eq.14 is able to be solved by the method of characteristics. Following the procedure of this method, a set of equivalent ordinary differential equations are obtained.

$$\frac{dt}{t}=\frac{d\boldsymbol{x}}{\boldsymbol{x}}=\frac{d\boldsymbol{u}}{\boldsymbol{u}}=\frac{dF}{2F}=\frac{dp}{2p}=\frac{dv}{2v} \tag{15}$$

Solving Eq.15 and regrouping the general solutions, the invariants under the action ofG_{s2} are determined.

$$C_1=\frac{x_i}{t\cdot u_i}=St, \qquad C_2=\frac{p}{u_i^2}=Eu, \qquad C_3=\frac{F}{u_i^2}=\frac{gx_i}{u_i^2}=Fr, \qquad C_4=\frac{u_ix_i}{v}=Re \tag{16}$$

where$C_j, (j=1,2,3,4)$ are the integration constants, and the subscript $'i'$ represents the three directions$(x,\ y,\ z)$. The above results show that the invariants conform to the four similitude criteria of wind tunnel tests. However, the concept of invariants considerably extends the mathematical and physical significance of the general similitude criteria.

In terms of deriving similitude criteria, Lie Group method presents many distinct aspects comparing to the traditional methods. By focusing on the symmetry properties of differential equations, this approach provides a powerful framework to derive all the transformations under which the NS equations still hold. These symmetric transformations are represented as symmetry Lie Groups. According to the symmetric mechanism presented by every symmetry group, an association of the scaling symmetry group of the NS equations with the scaled practices in wind tunnel tests is naturally built. Afterwards, a further derivation is implemented to obtain the invariants corresponding to the scaling symmetry group. From the unaltered property under scaling transformations, these invariants, appeared as integration constants, exactly conform to the definition of similitude criteria, and their expressions confirm to this inference.

The confliction between Fr number and Re number similitude criteria in a scaled model is well known, but the rigorous proof from the mathematical aspect has seldom been presented. Based on the above results, this work is easily accomplished.

GivenG_{s2}, the transformed variables are

$$\hat{\boldsymbol{y}}=\left(\hat{t},\hat{\boldsymbol{x}},\hat{\boldsymbol{u}},\hat{F},\hat{p},\hat{v}\right)=(t,e^{\epsilon}\boldsymbol{x},e^{\epsilon}\boldsymbol{u},e^{2\epsilon}F,e^{2\epsilon}p,e^{2\epsilon}v) \tag{17}$$

To simplify and consider the fact that gravity force is the dominant body force in the physically aeroelastic wind tunnel tests, take the vertical direction as an example.

$$\epsilon\ \frac{\partial w}{\partial t}+\epsilon u\frac{\partial w}{\partial x}+\epsilon v\frac{\partial w}{\partial y}+\epsilon w\frac{\partial w}{\partial z}=-\epsilon\frac{1}{\rho}\frac{\partial p}{\partial z}+\epsilon g+\epsilon v\frac{\partial^2 w}{\partial z^2} \tag{18}$$

From the mathematical point of view, it's obvious that the scaled NS equations, Eq.18, hold for any value of parameter ϵ. However, Eq.18 corresponds to the scaled physical phenomenon in wind tunnel tests, and parameter ϵ is in fact the chosen scale factor. The fact that the gravitational acceleration g is unable to be changed in most wind tunnel tests requires the scale factor ϵ being equal to 1. This inference is undoubtedly a sad requirement, since it means that only in the prototype can the Fr number and the Re number similitude criteria be satisfied simultaneously. The above investigation on the transformed the NS equations reveals the main reason for abandoning Re number similitude requirement in a vast majority of wind tunnel tests.

5 FLUTTER FORCE MODEL UNDER SCALED SYMMETRY TRANSFORMATION

As mentioned above, the semi-analytical method is widely adopted for investigating the aeroelastic performance of bridge structures. A majority of flutter force models mainly concerns the self-excited characteristic while relatively pay less attention to the wind aspects which are implicitly and coarsely considered through only several aeroelastic parameters. This indirect association makes the bridge structure and wind seem like two independent systems. Thus, there is a necessity to verify whether the scaled symmetry derived from the NS equations is compatible with the flutter force model.

Among several kinds of flutter force models, the one proposed by Scanlan is widely adopted. And the equations of motion for vertical and torsional degrees can be expressed as (Scanlan et al. 1971; Scanlan 1978),

$$m\ddot{h} + 2m\xi_h\omega_h\dot{h} + m\omega_h^2 h = \frac{1}{2}\rho U^2 B\left[KH_1^*\frac{\dot{h}}{U} + KH_2^*\frac{B\dot{\alpha}}{U} + K^2H_3^*\alpha + K^2H_4^*\frac{h}{B}\right] \tag{19a}$$

$$I\ddot{\alpha} + 2I\xi_\alpha\omega_\alpha\dot{\alpha} + I\omega_\alpha^2\alpha = \frac{1}{2}\rho U^2 B^2\left[KA_1^*\frac{\dot{h}}{U} + KA_2^*\frac{B\dot{\alpha}}{U} + K^2A_3^*\alpha + K^2A_4\frac{h}{B}\right] \tag{19b}$$

whereρ =air density; U =mean wind velocity; B =bridge width; $K = \omega B/U$ =reduced frequency; ω = circular frequency of motion; and H_i^* and $A_i^*(i = 1,2,3,4)$ = flutter derivatives.

According to the scaling infinitesimal generator of NS equations expressed in Eq.12, the vector field corresponding to Eq. 19 could be constructed under the similar scaling symmetric mechanism.

$$Y = \frac{\partial}{\partial t} + U\frac{\partial}{\partial U} - \frac{\partial}{\partial \omega} + B\frac{\partial}{\partial B} + 2m\frac{\partial}{\partial m} + 4I\frac{\partial}{\partial I} \tag{20}$$

The invariants of this vector field are assumed to be η, and it could be solved from following equation.

$$Y(\eta) = \frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial U} - \frac{\partial \eta}{\partial \omega} + B\frac{\partial \eta}{\partial B} + 2m\frac{\partial \eta}{\partial m} + 4I\frac{\partial \eta}{\partial I} = 0 \tag{21}$$

With the similar procedures conducted in the previous section, invariants which guarantee the scaling symmetric mechanism derived from the NS equations compatible with the flutter force model are determined.

$$C_5 = \frac{\omega B}{U} = K, \qquad C_6 = \frac{m}{B^2}, \qquad C_7 = \frac{I}{B^4} \tag{22}$$

Clearly, the first derived invariant C_5 is consistent with the reduced frequency. This result endows the routinely defined reduced frequency in bridge flutter analysis a mathematical explanation that the essence of reduced frequency is an invariant under the scaling transformation. The other two invariants represent the ratios between the structural masses and the corresponding equivalent air mass. These mean that only do the mass ratios keep consistent, can the scaling symmetry be compatible with the one of the NS equations.

6 CONCLUSIONS

This paper proposed a new method to derive the similitude criteria of scaled bridge wind tunnel tests. As Lie Group is a powerful tool for solving the symmetry transformations of the NS equations, this method is more generalized comparing to the traditional methods. Furthermore, the extension to the concept of invariants renders a mathematical explanation to the similitude criteria and makes a rigorous proof of the confliction between Fr Number and Re number in a scaled model possible. The successful application of this method to the classical flutter force model not only verifies the compatibility of scaling symmetric mechanism between the fluid and the structural system, but also determines the invariants of the structural system. Based on the solved invariants, the invariant essence of the reduced frequency is revealed. Summarily, Lie symmetry group method removes the abstract aspects of the similitude theory but provides a powerful mathematical platform for deriving the similitude criteria.

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