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A nonlocal model for bending of nanobeams

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ABSTRACT

An enhanced Euler-Bernoulli nanobeam model is presented. The coupled nonlocal model depends on two nonlocal parameters and is obtained by using a suitable definition of the free energy. As special cases, nanobeam formulations based on the Eringen nonlocal elasticity theory and on the gradient elastic model can be recovered from the proposed coupled model. A closed-form solution and a numerical example are presented to illustrate the versatility and efficiency of the proposed nonlocal model.

1. INTRODUCTION

The classical (local) Euler–Bernoulli beam theory assumes that straight lines normal to the midplane before deformation remain straight and normal to the midplane after deformation.

A nonlocal Euler-Bernoulli beam model has been introduced by Peddieson et al. (2003) in which the nonlocal model is obtained by replacing the stress appearing in the classical equilibrium equations by its nonlocal counterpart. Such an approach has been widely followed by many researchers, see e.g. Reddy (2007), Wang and Liew. (2007), Aydogdu (2009), Arash and Wang (2012). Moreover, a hybrid nonlocal beam model is developed in Zhang et al. (2010) by postulating that the strain energy involves both local and nonlocal curvatures. Accordingly, the hybrid model shows nonlocal effects for an Euler-Bernoulli cantilever nanobeam under a transverse point load while the Eringen model is free of small-scale effects for the same problem (Challamel and Wang, 2008).

In the present contribution, a new coupled nonlocal model is introduced by a suitable definition of the free energy which depends on two small length-scale parameters. The related variational formulation can then be built up by adopting a consistent methodology based on nonlocal thermodynamics. Therefore, the differential relations with the associated boundary conditions can be obtained in a straightforward manner. A suitable choice of the two small length-scale parameters appearing in the nonlocal model can make the nanobeam flexible or stiffer.

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The proposed coupled model can be specialized to recover the Eringen model, Eringen (1983), (1987), Barretta R., Marotti de Sciarra (2013), and the gradient model Aifantis (2003), (2014), Papargyri-Beskou et al. (2003), Giannakopoulos and Stamoulis (2007), Akgöz and Civalek (2012).

An example of a nanocantilever subjected to a uniform load is illustrated and the closed-form solution for the considered models is provided in order to investigate the influence of the nonlocal parameters. A comparison among the coupled, Eringen and gradient model is thus performed.

2. NONLOCAL ELASTIC MODELS

An Euler-Bernoulli straight nanobeam occupying a domain *V* is considered. The cross-section of the nanobeam is denoted by *A*, the centroid axis is indicated by *x* and the bending plane is defined by the Cartesian axes *(x, y)* originating at the cross-section centroid. The *z*-axis is orthogonal to the bending plane.

The displacement field of the nanobeam and the kinematically compatible axial deformation field are

$$\boldsymbol{s}(x,y,z)=\begin{bmatrix}-v^{(1)}(x)y\\ v(x)\\ 0\end{bmatrix},\qquad \varepsilon(x,y,z)=-v^{(2)}(x)y \tag{1}$$

where v is the transverse displacement along the *y*-axis and $v^{(2)}$ is the nanobeam bending curvature. The apex denotes the derivative along the nanobeam axis *x*.

The first principle of thermodynamic can be written in a global form and the second one can be expressed in its usual local form (Polizzotto, 2003; Marotti de Sciarra 2009, Romano et al. 2010) so that the vanishing of the body energy dissipation is expressed as

$$\int_V \bar{\sigma}\dot{\varepsilon}\,dV=\int_V \dot{\psi}\,dV \tag{2}$$

where ψ is the Helmholtz free energy of the nanobeam, $\bar{\sigma}$ is the nonlocal axial stress and ε is the axial strain. The superscript dot denotes differentiation with respect to the time.

Three nonlocal models for nanobeams are addressed by considering the following three different expressions of the free energy ψ_i

$$i)\ \psi_1(\varepsilon)=\frac{1}{2}E\varepsilon^2+c^2\bar{\sigma}^{(2)}\varepsilon \tag{3}$$

$$ii)\ \psi_2\left(\varepsilon,\varepsilon^{(1)}\right)=\frac{1}{2}E\varepsilon^2+\frac{1}{2}c^2E\varepsilon^{(1)2} \tag{4}$$

and

$$iii)\ \psi_3\left(\varepsilon,\varepsilon^{(1)}\right)=\frac{1}{2}E\varepsilon^2+\frac{1}{2}c^2E\varepsilon^{(1)2}+\alpha c^2\bar{\sigma}^{(2)}\varepsilon. \tag{5}$$

The Young modulus of the material is E and the length-scale parameter is c. The dimensionless parameter α acts as a participation factor so that the model iii) depends on two nonlocal parameters c and α. The mechanical meaning of the parameter α and the link between the above nonlocal models and existing theories will be clarified in the sequel.

Note that the participation factor α multiples the nonlocal part of the free energy ψ_1 and the free energy ψ_3 reduces to ψ_2 by setting $\alpha = 0$ so that the free energy (5) can be considered as a coupling between the free energies ψ_1 and ψ_2.

Substituting the time derivative of the free energy given by Eqs. (3), (4) and (5) into Eq. (2) and using the kinematically compatible deformation field $(1)_2$, the following variational formulations associated with the models i), ii) and iii) are obtained

$$i)\ \int_0^L M\dot{v}^{(2)}\,dx = \int_0^L M_0\dot{v}^{(2)}\,dx + c^2\int_0^L M^{(2)}\dot{v}^{(2)}\,dx \tag{6}$$

$$ii)\ \int_0^L M\dot{v}^{(2)}\,dx = \int_0^L M_0\dot{v}^{(2)}\,dx + c^2\int_0^L M_1\dot{v}^{(3)}\,dx \tag{7}$$

and

$$iii)\ \int_0^L M\dot{v}^{(2)}\,dx = \int_0^L M_0\dot{v}^{(2)}\,dx + \alpha c^2\int_0^L M^{(2)}\dot{v}^{(2)}\,dx + c^2\int_0^L M_1\dot{v}^{(3)}\,dx \tag{8}$$

where the stress resultant moments are given by

$$(M, M_0, M_1) = -\int_A (\bar{\sigma}, \sigma_0, \sigma_1)\,ydA = -\int_A \left(\bar{\sigma}, E\varepsilon, E\varepsilon^{(1)}\right) ydA. \tag{9}$$

The variational formulation (6) provides the elastic nonlocal model proposed by Eringen (1972), (1983), (1987) and the variational formulation (7) yields the nonlocal gradient elasticity model (Papargyri-Beskou, 2003; Akgöz and Civalek, 2012; Marotti de Sciarra 2014). It is worth noting that these models are usually recovered following a different path of reasoning. In fact, the starting point is the definition of the nonlocal stress which is commonly expressed in the Eringen form. Hence, the stress in the classical equilibrium equations is replaced by the corresponding nonlocal quantity and the nonlocal model is thus derived. On the contrary, the proposed approach starts from the definition of the free energy so that the related variational formulation can be consistently provided and the nonlocal model is thus obtained.

Following the suggested approach, the new expression of the free energy (5) leads to the new variational formulation (8) for nonlocal Euler-Bernoulli nanobeams and the corresponding nonlocal model can be considered as a combination between the nonlocal Eringen and gradient models as shown in the sequel.

Integrating by parts Eqs. (6), (7) and (8), the following nonlocal differential relations are provided

$$i)\ M^{(2)} - c^2 M^{(4)} = M_0^{(2)} \tag{10}$$

$$ii)\ M^{(2)} = M_0^{(2)} - c^2 M_1^{(3)} \tag{11}$$

and

$$iii)\; M^{(2)} - \alpha c^2 M^{(4)} = M_0^{(2)} - c^2 M_1^{(3)} \tag{12}$$

where the corresponding boundary conditions consistently follow from the related variational principle and are reported in Table 1.

Table 1 - Boundary conditions for the nanobeam models

Nonlocal Model	Kinematic boundary conditions	Static boundary conditions
i)	v	$-M^{(1)} + c^2 M^{(3)} = -M_0^{(1)}$
	$v^{(1)}$	$M - c^2 M^{(2)} = M_0$
ii)	v	$-M^{(1)} = -M_0^{(1)} + c^2 M_1^{(2)}$
	$v^{(1)}$	$M = M_0 - c^2 M_1^{(1)}$
	$v^{(2)}$	$0 = c^2 M_1$
iii)	v	$-M^{(1)} + \alpha c^2 M^{(3)} = -M_0^{(1)} + c^2 M_1^{(2)}$
	$v^{(1)}$	$M - \alpha c^2 M^{(2)} = M_0 - c^2 M_1^{(1)}$
	$v^{(2)}$	$0 = c^2 M_1$

The classical (local) differential relationship between bending moment and applied load is recovered for the nonlocal models i), ii) and iii) by considering the l.h.s. of Eq. (2). Replacing $\dot{\varepsilon}$ by using the kinematically compatible relation in Eq. $(1)_2$ and integrating by parts it results

$$M^{(2)} = q. \tag{13}$$

The boundary conditions at $x = \{0, L\}$ provide the conditions $T = -M^{(1)} = F$ and $M = \mathcal{M}$ where *T* is the shear force, *q* is the distributed transverse load and $(F, \mathcal{M})$ are the transverse force and couple respectively.

The nonlocal elastic equilibrium equation for nanobeams associated with the considered models can then be provided by expressing the differential equations (10), (11) and (12) in terms of the transverse displacement *v* using Eqs. (1) and (9). In fact, setting $I = \int_A y^2 dA$ the second moment of area about the *z*-axis and noting the equalities

$$(M_0, M_1) = -\int_A \left(E\varepsilon, E\varepsilon^{(1)}\right) y dA = \left(EIv^{(2)}, EIv^{(3)}\right), \tag{14}$$

the governing differential equations for the bending of the nonlocal Euler-Bernoulli nanobeam under distributed transverse loads are

$$i)\; EIv^{(4)} = q - c^2 q^{(2)} \tag{15}$$

$$ii)\; c^2 EIv^{(6)} - EIv^{(4)} = -q \tag{16}$$

and

$$iii)\ c^2 EIv^{(6)} - EIv^{(4)} = -q + \alpha c^2 q^{(2)}. \tag{17}$$

The boundary conditions follow from Table 1 and are reported in Table 2.

Table 2 Boundary conditions in terms of transverse displacement

Nonlocal Model	Kinematic boundary conditions	Static boundary conditions
i)	v	$-EIv^{(3)} = T + c^2 q^{(1)}$
	$v^{(1)}$	$EIv^{(2)} = M - c^2 q$
ii)	v	$-EIv^{(3)} + c^2 EIv^{(5)} = T$
	$v^{(1)}$	$EIv^{(2)} - c^2 EIv^{(4)} = M$
	$v^{(2)}$	$c^2 EIv^{(3)} = 0$
iii)	v	$-EIv^{(3)} + c^2 EIv^{(5)} = T + \alpha c^2 q^{(1)}$
	$v^{(1)}$	$EIv^{(2)} - c^2 EIv^{(4)} = M - \alpha c^2 q$
	$v^{(2)}$	$c^2 EIv^{(3)} = 0$

The free energy (3) provides the fourth-order differential equation (15) governing the bending of the Euler-Bernoulli nanobeam according to the nonlocal model proposed by Eringen, see e.g. Wang and Liew (2007). In order to solve Eq. (15), four boundary conditions are required (see Table 2). Note that the nonlocal parameter enters only in the boundary conditions if the distributed transverse load is null, constant or linear.

The coupled model ii) provides the Euler-Bernoulli nanobeam associated with the gradient elasticity model which is governed by the sixth-order differential equation (16) and the boundary conditions are reported in Table 2 - ii).

The free energy (5) yields the sixth-order differential equations (17) governing the bending of the Euler-Bernoulli nanobeam. Accordingly, six boundary conditions (three for each end of the nanobeam) are required (see Table 2 - iii) and the length-scale parameter appears in the differential equations as well as in the boundary conditions.

The differential equation (17) of the coupled model iii) and the related boundary conditions in Table 2 – iii) encompass terms following from the Eringen and gradient models.

The coupled model iii) provides the Euler-Bernoulli nanobeam associated with the gradient elasticity model if $\alpha = 0$.

The expression of the bending moment for the three models is recovered from the related variational formulations (6), (7) and (8). In fact, integrating by parts the last integrals into Eqs. (7) and (8) and recalling that the boundary conditions in Table 1 ensures $M_1 = 0$ for both models ii) and iii), we get

$$i)\ M = M_0 + c^2 q = EIv^{(2)} + c^2 q \tag{18}$$

$$ii)\ M = M_0 - c^2 M_1^{(1)} = EIv^{(2)} - c^2 EIv^{(4)} \tag{19}$$

and

$$iii)\; M = M_0 - c^2 M_1^{(1)} + \alpha c^2 q = EIv^{(2)} - c^2 EIv^{(4)} + \alpha c^2 q. \tag{20}$$

The considered nonlocal models tends to the classical (local) one for a vanishing nonlocal parameter c. Moreover an upper bound for the displacement field in terms of the nonlocal parameter c can be evaluated for the gradient model ii) as shown in the next Section 3.

3. CLOSED-FORM SOLUTIONS FOR A NANOCANTILEVER

A nanocantilever with length L subjected to a uniform load q is considered. The nonlocal solution pertaining to the Eringen model i) is obtained by solving the fourth-order differential equation (15) with the following boundary conditions

$$i)\; \begin{cases} v(0) = 0 \\ v^{(1)}(0) = 0 \\ v^{(3)}(L) = 0 \\ EIv^{(2)}(L) = -c^2 q. \end{cases} \tag{21}$$

Note that the nonlocal parameter c enters only in the last boundary condition.

The nonlocal solution of the gradient model ii) is obtained by solving the sixth-order differential equation (16) with the following six boundary conditions

$$ii)\; \begin{cases} v(0) = 0 \\ v^{(1)}(0) = 0 \\ v^{(3)}(0) = 0 \\ v^{(3)}(L) - c^2 v^{(5)}(L) = 0 \\ v^{(2)}(L) - c^2 v^{(4)}(L) = 0 \\ v^{(3)}(L) = 0. \end{cases} \tag{22}$$

The nonlocal solution of the coupled model iii) is achieved by solving the sixth-order differential equation (17) with the following six boundary conditions

$$iii)\; \begin{cases} v(0) = 0 \\ v^{(1)}(0) = 0 \\ v^{(3)}(0) = 0 \\ v^{(3)}(L) - c^2 v^{(5)}(L) = 0 \\ EIv^{(2)}(L) - c^2 EIv^{(4)}(L) = -\alpha c^2 q \\ v^{(3)}(L) = 0. \end{cases} \tag{23}$$

Hence, the transverse displacement field of the Eringen model is obtained in the following form

$$i)\ v_1(x) = v_0(x) - c^2 \frac{qx^2}{2EI}, \qquad \text{with } v_0(x) = \frac{qx^4}{24EI} - \frac{qLx^3}{6EI} + \frac{qL^2x^2}{4EI} \tag{24}$$

where v_0 is the classical (local) transverse displacement.

The transverse displacement field of the gradient model is

$$ii)\ v_2(x) = v_0(x) + \frac{c^3\left(1+e^{\frac{2L}{c}}\right)qL}{\left(-1+e^{\frac{2L}{c}}\right)EI} + \frac{c^3 e^{\frac{x}{c}}qL}{\left(1-e^{\frac{2L}{c}}\right)EI} + \frac{c^3 e^{\frac{2L}{c}-\frac{x}{c}}qL}{\left(1-e^{\frac{2L}{c}}\right)EI} - \frac{c^2qLx}{EI} + \frac{2qc^2x^2}{4EI} \tag{25}$$

and the transverse displacement field of the coupled model is

$$iii)\ v_3(x) = v_0(x) + \frac{c^3\left(1+e^{\frac{2L}{c}}\right)qL}{\left(-1+e^{\frac{2L}{c}}\right)EI} + \frac{c^3 e^{\frac{x}{c}}qL}{\left(1-e^{\frac{2L}{c}}\right)EI} + \frac{c^3 e^{\frac{2L}{c}-\frac{x}{c}}qL}{\left(1-e^{\frac{2L}{c}}\right)EI} - \frac{c^2qLx}{EI} - \frac{2qx^2c^2(-1+\alpha)}{4EI}. \tag{26}$$

The transverse displacement v_1, v_2, and v_3 reduce to the classical (local) one v_0 for $c = 0$.

The upper bound v_∞ of the displacement field in terms of the scale parameter c for the gradient model can be obtained by evaluating the limit of v_2 for $c \to +\infty$. Hence, the displacement field of the gradient model must belong to the strip bounded by the functions:

$$v_0(x) = \frac{qx^4}{24EI} - \frac{qLx^3}{6EI} + \frac{qL^2x^2}{4EI} \qquad and \qquad v_\infty(x) = \frac{qL^2x^2}{12EI}. \tag{27}$$

The bending moment of the three models is obtained by substituting Eqs. (24), (25) and (26) into Eqs. (18), (19) and (20). For all the considered models, the expressions of the bending moment coincide to the classical (local) one $q(L-x)^2/2$.

3.1 Example - nanocantilever under a uniform load

It is convenient to introduce the following dimensionless quantities:

$$\begin{aligned} &\xi = \frac{x}{L}, \qquad \eta = \frac{y}{L}, \qquad \tau = \frac{c}{L}, \\ &v_i^*(\xi) = v_i(x)\frac{EI}{qL^4}, \end{aligned} \tag{28}$$

with $i = \{1,2,3\}$ so that only the following material properties are required in computations $\tau = \{0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.4, 0.5\}$ and $\alpha = \{-3.0, -2.0, 0.1, 0.5, 1.0\}$.

The dimensionless transverse displacements of the nanocantilever under a uniform load for the considered three models and for different values of τ and α are reported in Figs. 1(a)-(b).

The maximum deflection of the nanocantilever occurs at the free end-section $\xi = 1$ but such a deflection, evaluated using the Eringen model i), decreases for increasing τ and vanishes for $\tau = 0.5$, see Figs. 1(a)-(b).

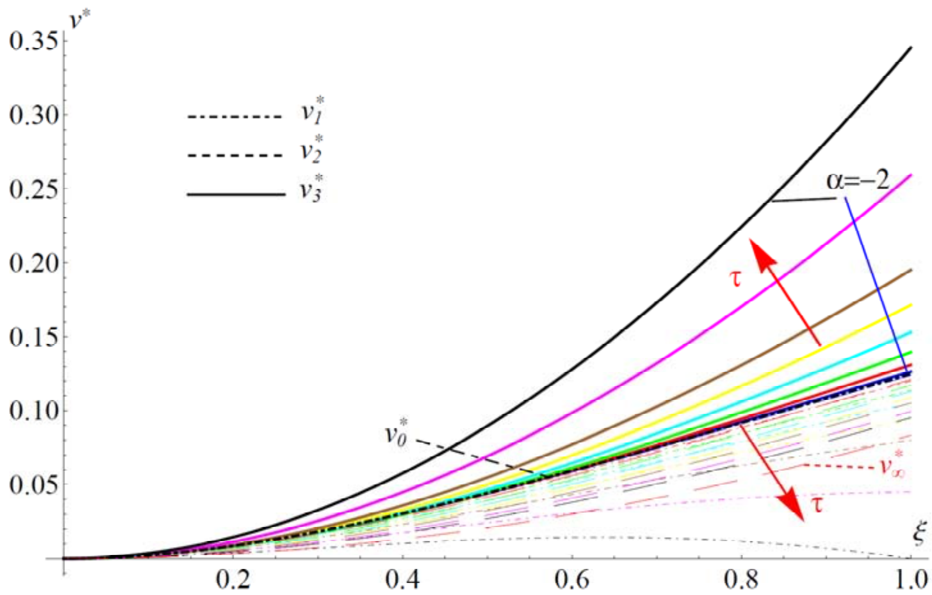

Fig. 1(a) Dimensionless transverse displacements of the nanocantilever under a uniform load. Comparison among the Eringen model v_1^*, gradient model v_2^* and coupled model v_3^* with $\alpha = -2$.

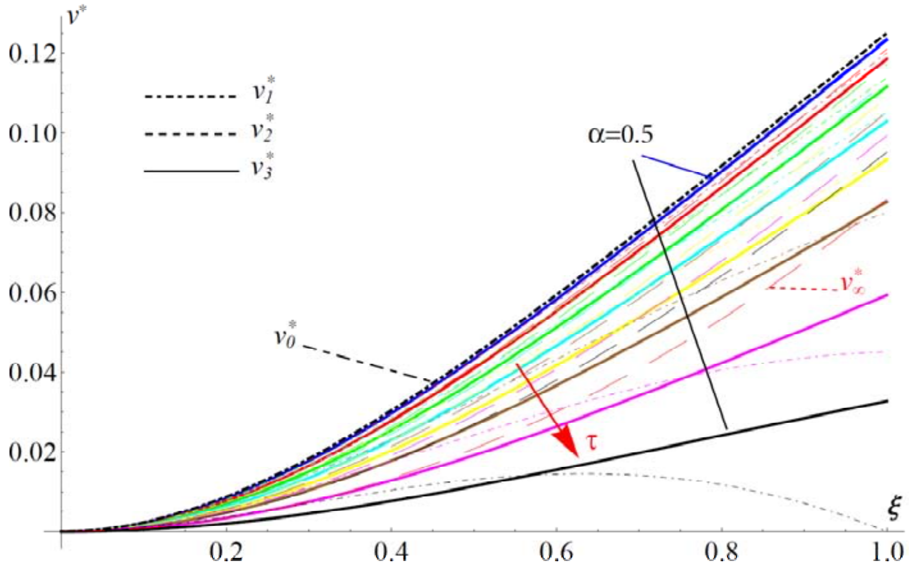

Fig. 1(b) Dimensionless transverse displacements of the nanocantilever under a uniform load. Comparison among the Eringen model v_1^*, gradient model v_2^* and coupled model v_3^* with $\alpha = 0.5$.

The dimensionless transverse displacement pertaining to the coupled model iii) is plotted for the considered values of τ with $\alpha = -2.0$ in Fig. 1(a) and with $\alpha = 0.5$ in Fig. 1(b).

It is apparent that the nanocantilever softens with increasing τ for a negative participation factor α. On the contrary, the nanocantilever stiffens with increasing τ if the parameter α is positive.

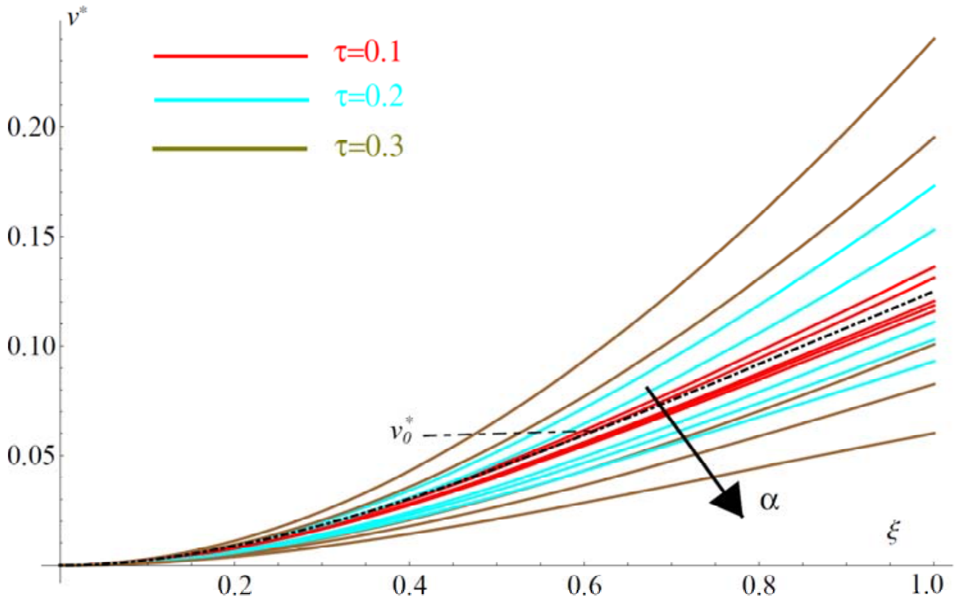

Fig. 2 Dimensionless transverse displacements v_3^* of the nanocantilever under a uniform load for increasing values of the participation factor α.

Fig. 2 shows the dimensionless transverse displacements v_3^* of the nanocantilever for $\tau = \{0.1, 0.2, 0.3\}$ and $\alpha = \{-3.0, -2.0, 0.1, 0.5, 1.0\}$. It is apparent that the deflection of the nanocantilever increases with increasing τ if the parameter α is negative and decreases with increasing τ if α is positive.

4. CONCLUSIONS

A new coupled nonlocal Euler-Bernoulli nanobeam model is presented based on a consistent thermodynamic approach. The governing equations and the related high-order boundary conditions are derived by using a variational formulation associated with the nonlocal model.

The proposed coupled model encompasses the gradient elastic beam model and the Eringen model as special cases and reduces to the classical (local) Euler-Bernoulli beam model for a vanishing small-scale parameter.

Closed-form solutions for the bending problem of a nanocantilever under a distributed load are presented and numerical results are compared with the one obtained using the gradient elastic and Eringen models.

The participation factor introduced in the coupled model can make the nanobeam flexible or stiffer when compared the gradient elastic model, the Eringen model and the classical (local) beam model.

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