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Effect of white layer on the distribution of residual stress in thin discs

*Sergei Alexandrov[1)], Yusuf Mustafa[2)], Zaini Ahmad[2)] and Nguyen Kien[3)]

[1)] Laboratory of Fracture Mechanics, Institute for Problems in Mechanics, Moscow 11926, Russia
[1)] sergei_alexandrov@spartak.ru
[2)] Department of Mechanical Engineering, Universiti Teknologi Malaysia, Johor Bahru, 81310, Malaysia
[2)] mustafa51059@gmail.com
[3)] Department of Sold Mechanics, Institute of Mechanics, Hanoi 305-600, Vietnam
[3)] ndkien@imech.ac.vn

ABSTRACT

The Mises yield criterion and its associated flow rule are adopted to provide a semi-analytic solution for the distribution of residual stresses within a thin hollow disc inserted into a rigid container and subject to thermal loading by a uniform temperature field and subsequent unloading. It is assumed that there is a narrow hard layer in the vicinity of the hole generated by a preceding treatment of the surface. It follows from available experimental data that the elastic modulus and yield stress within this layer are much higher than the elastic modulus and yield stress of the base material. This paper examines the effect of this difference in the mechanical properties on the distribution of residual stresses. The primary objective of the paper is to provide a benchmark problem having a semi-analytic solution for justifying the possibility to neglect or the necessity to account for the presence of narrow hard layers in analysis of elastic-plastic discs under plane stress conditions. Numerical techniques are only necessary to evaluate ordinary integrals and solve an ordinary differential equation.

1. INTRODUCTION

"White layer" is a term referring to hard layers of material in the vicinity of surfaces that are generated during various machining and deformation processes (Griffiths, 1987). The majority of white layer publications has been concerned with mechanisms of the generation of such layers and wear (Griffiths, 1987, Cho et al, 2012, Huang et al, 2013 among many others). Influence of hard layers on the development of rolling contact fatigue has been demonstrated by Warren and Guo (2005) using a numerical

[1)] Professor
[2)] Professor
[3)] Professor

method and by Choi (2010) using an experimental technique. It is therefore of interest to understand how hard layers affect structure and component performance under other loading conditions. In particular, it has been found in Cho et al (2012) that the elastic modulus and yield stress within white layers may increase by 170% and 390%, respectively. It is therefore reasonable to expect that such a huge difference in the mechanical properties between the narrow surface layer and base material affects the distribution of stresses and strains including residual stresses and strains in the structure under service conditions. Analytic and semi-analytic solutions are very useful to reveal this possible effect, even though such solutions by necessity involve simplifying assumptions. The solution given in the present paper deals with a hollow disc inserted into a container and subject to thermal loading and subsequent unloading. This is an ideal boundary value problem to study various qualitative features of solutions for elastic/plastic discs under plane stress conditions (Alexandrov and Alexandrova, 2001, Alexandrov et al, 2012, Alexandrov et al, 2014[a,b]). The distinguished feature of the problem considered in the present paper is that the presence of a hard layer in the vicinity of the hole is taken into account.
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2. STATEMENT OF THE PROBLEM

Consider a thin hollow elastic-plastic disc with inner and outer radii a_0 and R_0, respectively inserted into a rigid container of radius R_0 and subjected to thermal loading by an uniform temperature field. There is a narrow hard layer of material in the vicinity of the inner radius generated by a previous machining or deformation processes. The outer radius of this layer is b_0. The disc has no stress at the initial instant. The increase in temperature from its initial value, *T*, and the constraints imposed on the disc affects the zero stress state. It is natural to introduce a cylindrical coordinate system (r, θ, z) with its *z*- axis coinciding with the axis of symmetry of the disc. Symmetry dictates that the normal stresses in this coordinate system, σ_r, σ_θ and σ_z, are the principal stresses and the circumferential displacement vanishes everywhere. The radial displacement is denoted by *u*. It is supposed that the state of stress is plane ($\sigma_z = 0$) and the strains are small. Plastic yielding is controlled by the Mises yield criterion. In the case under consideration this criterion can be written as

$$\begin{aligned} \sigma_r^2 + \sigma_\theta^2 - \sigma_\theta\sigma_r &= \sigma_0^2 \quad \text{in the range} \quad b_0 \le r \le R_0, \\ \sigma_r^2 + \sigma_\theta^2 - \sigma_\theta\sigma_r &= \sigma_H^2 \quad \text{in the range} \quad a_0 \le r \le b_0. \end{aligned} \tag{1}$$

Here σ_0 and σ_H are the tensile yield stresses of the base material and the hard layer, respectively. Both σ_0 and σ_H are material constants. The classical Duhamel-Neumann law is adopted. In particular, the elastic portions of the total strains are related to the stresses as

$$\varepsilon_r^e = \frac{\sigma_r - \nu\sigma_\theta}{E}, \quad \varepsilon_\theta^e = \frac{\sigma_\theta - \nu\sigma_r}{E}, \quad \varepsilon_z^e = -\frac{\nu(\sigma_r + \sigma_\theta)}{E} \tag{2}$$

where *E* is Young's modulus and ν is Poisson's ratio. It is assumed that $E = E_0$ in the range $b_0 \le r \le R_0$ and $E = E_H$ in the range $a_0 \le r \le b_0$. The thermal portions of the total strains are given by

$$\varepsilon_r^T = \varepsilon_\theta^T = \varepsilon_z^T = \alpha T \tag{3}$$

where α is the thermal coefficient of linear expansion. Both ν and α are independent of *r*. The total strains in plastic regions are

$$\varepsilon_r = \varepsilon_r^T + \varepsilon_r^e + \varepsilon_r^p, \quad \varepsilon_\theta = \varepsilon_\theta^T + \varepsilon_\theta^e + \varepsilon_\theta^p, \quad \varepsilon_z = \varepsilon_z^T + \varepsilon_z^e + \varepsilon_z^p \tag{4}$$

where $\varepsilon_r^p, \varepsilon_\theta^p$ and ε_z^p are the plastic portions of the total strains. In the case under consideration, the total radial and circumferential strains are

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}. \tag{5}$$

The flow theory of plasticity is adopted. Therefore, the associated flow rule connects stresses and strain rates rather than strains. Since the strains are small, the components of the strain rate tensor are obtained as the local time derivatives of the corresponding components of the strain tensor. However, since the material model is rate-independent, these time derivatives can be replaced with the corresponding derivatives with respect to any other monotonically increasing parameter. Denote this parameter by *p*. Then,

$$\begin{gathered}
\zeta_r = \frac{\partial \varepsilon_r}{\partial p}, \quad \zeta_\theta = \frac{\partial \varepsilon_\theta}{\partial p}, \quad \zeta_z = \frac{\partial \varepsilon_z}{\partial p}, \\
\zeta_r^e = \frac{\partial \varepsilon_r^e}{\partial p}, \quad \zeta_\theta^e = \frac{\partial \varepsilon_\theta^e}{\partial p}, \quad \zeta_z^e = \frac{\partial \varepsilon_z^e}{\partial p}, \\
\zeta_r^p = \frac{\partial \varepsilon_r^p}{\partial p}, \quad \zeta_\theta^p = \frac{\partial \varepsilon_\theta^p}{\partial p}, \quad \zeta_z^p = \frac{\partial \varepsilon_z^p}{\partial p}.
\end{gathered} \tag{6}$$

The associated flow rule gives

$$\zeta_r^p = \lambda(2\sigma_r - \sigma_\theta), \quad \zeta_\theta^p = \lambda(2\sigma_\theta - \sigma_r), \quad \zeta_z^p = -\lambda(\sigma_r + \sigma_\theta) \tag{7}$$

where $\lambda \ge 0$. It is convenient to introduce the following dimensionless quantities

$$\rho = \frac{r}{R_0}, \quad a = \frac{a_0}{R_0}, \quad b = \frac{b_0}{R_0}, \quad \gamma = \frac{\sigma_0}{\sigma_H}, \quad \beta = \frac{E_0}{E_H}, \quad k = \frac{\sigma_0}{E_0}, \quad k_H = \frac{\sigma_H}{E_H} = k\frac{\beta}{\gamma},$$
$$\tau = \frac{\alpha T}{k}, \quad \tau_H = \frac{\alpha T}{k_H} = \tau\frac{\gamma}{\beta}. \tag{8}$$

The only non-trivial equilibrium equation is

$$\frac{\partial \sigma_r}{\partial \rho} + \frac{\sigma_r - \sigma_\theta}{\rho} = 0\,. \tag{9}$$

The boundary conditions are

$$\sigma_r = 0 \tag{10}$$

for $\rho = a$ and $u = 0$ for $\rho = 1$. It is evident from Eq. (5) that the latter is equivalent to

$$\varepsilon_\theta = 0 \tag{11}$$

for $\rho = 1$. In addition, the radial stress and the radial displacement must be continuous across the surface $\rho = b$. Therefore,

$$\left[\sigma_r\right] = 0 \quad \text{and} \quad \left[\varepsilon_\theta\right] = 0 \tag{12}$$

for $\rho = b$. Here $[\ \ ]$ denotes the amount of jump in the quantity in the brackets.

3. PURELY ELASTIC SOLUTION

The general thermo-elastic solution is well known. Using Eq.(8) this solution is written as

$$\frac{\sigma_r}{\sigma_0} = \frac{A}{\rho^2} + B, \qquad \frac{\sigma_\theta}{\sigma_0} = -\frac{A}{\rho^2} + B\,, \tag{13}$$

$$\varepsilon_r = k\left[(1+\nu)\frac{A}{\rho^2} + (1-\nu)B + \tau\right], \quad \varepsilon_\theta = k\left[(1-\nu)B - (1+\nu)\frac{A}{\rho^2} + \tau\right], \quad \varepsilon_z = -k(2\nu B - \tau) \tag{14}$$

in the range $b \le \rho \le 1$ and

$$\frac{\sigma_r}{\sigma_H} = \frac{A_H}{\rho^2} + B_H, \qquad \frac{\sigma_\theta}{\sigma_H} = -\frac{A_H}{\rho^2} + B_H\,, \tag{15}$$

$$\varepsilon_r = k_H\left[(1+\nu)\frac{A_H}{\rho^2}+(1-\nu)B_H+\tau_H\right],\quad \varepsilon_\theta = k_H\left[(1-\nu)B_H-(1+\nu)\frac{A_H}{\rho^2}+\tau_H\right],\quad \varepsilon_z=-k_H\left(2\nu B_H-\tau_H\right) \tag{16}$$

in the range $a\le\rho\le b$ where *A*, *B*, A_H and B_H are constants of integration. Substituting Eq.(15) into Eq.(10) and Eq.(14) into Eq.(11) yield

$$A_H=-B_H a^2 \tag{17}$$

and

$$(1-\nu)B-(1+\nu)A+\tau=0. \tag{18}$$

It follows from Eqs. (8), (12) - (16) that

$$\left(A+b^2B\right)\gamma=A_H+b^2B_H,\quad (1-\nu)Bb^2-(1+\nu)A=\frac{\beta}{\gamma}\left[(1-\nu)B_Hb^2-(1+\nu)A_H\right]. \tag{19}$$

Eliminating here A_H and *A* by means of Eqs.(17) and (18) gives

$$\gamma\left(b^2+\frac{1-\nu}{1+\nu}\right)B-\left(b^2-a^2\right)B_H=-\frac{\gamma\tau}{1+\nu},\quad (1-\nu)\left(b^2-1\right)B-\frac{\beta}{\gamma}\left[(1-\nu)b^2+(1+\nu)a^2\right]B_H=\tau. \tag{20}$$

Solving this system for *B* and B_H and, then, using Eqs.(17) and (18) allow for determining the distribution of stress and strain by means of Eqs.(13) – (16). Substituting Eqs.(13) and (15) into Eq.(1) shows that the purely elastic solution is valid if

$$B^2+\frac{3A^2}{\rho^4}\le 1 \quad\text{and}\quad B_H^2+\frac{3A_H^2}{\rho^4}\le 1. \tag{21}$$

It is evident from these inequalities that plastic yielding may start either in the hard layer at $\rho=a$ or in the base material at $\rho=b$. These conditions for the initiation of plastic yielding are written as

$$B^2+\frac{3A^2}{b^4}=1 \quad\text{and}\quad B_H^2+\frac{3A_H^2}{a^4}=1. \tag{22}$$

For a given configuration and given material properties, *A*, *B*, A_H and B_H depend on τ. Solving the equations shown in Eq. (22) for τ determines two critical values, τ_b and τ_h. Here τ_b is the solution to Eq.(22)[1] and τ_h to Eq.(22)[2]. The initiation of plastic yielding occurs at $\rho = a$ if $\tau_h < \tau_b$ and at $\rho = b$ if $\tau_h > \tau_b$. Two plastic zones start to develop simultaneously if $\tau_h = \tau_b$.

4. ELASTIC/PLASTIC SOLUTION

Since $E_H > E_0$ and $\sigma_H > \sigma_0$, it is reasonable to expect that $\tau_h > \tau_b$ if $b_0 - a_0$ is small enough. This assumption should be verified *a posteriori*. There are two elastic zones, $a \le \rho \le b$ and $\rho_c \le \rho \le 1$, and the plastic zone $b \le \rho \le \rho_c$. Here ρ_c is the dimensionless radius of the elastic/plastic boundary. The yield criterion (1) is satisfied by the following substitution

$$\frac{\sigma_r}{\sigma_0} = \frac{2}{\sqrt{3}} \sin\left(\psi + \frac{\pi}{3}\right), \quad \frac{\sigma_\theta}{\sigma_0} = -\frac{2}{\sqrt{3}} \sin\left(\psi - \frac{\pi}{3}\right) \tag{23}$$

where ψ is a new unknown function of *p* and ρ. Substituting Eq.(23) into Eq.(9) and using Eq.(8) give

$$\rho \cos\left(\psi + \frac{\pi}{3}\right) \frac{\partial \psi}{\partial \rho} + \sin\psi = 0. \tag{24}$$

Let ψ_b be the value of ψ at $\rho = b$. Solving Eq.(24) with the use of the boundary condition $\psi = \psi_b$ for $\rho = b$ results in

$$\frac{\rho}{b} = \sqrt{\frac{\sin\psi_b}{\sin\psi}} \exp\left[\frac{\sqrt{3}}{2}\left(\psi - \psi_b\right)\right]. \tag{25}$$

Let ψ_c be the value of ψ at the elastic/plastic boundary. Then, it follows from Eq.(25) that

$$\frac{\rho_c}{b} = \sqrt{\frac{\sin\psi_b}{\sin\psi_c}} \exp\left[\frac{\sqrt{3}}{2}\left(\psi_c - \psi_b\right)\right]. \tag{26}$$

The elastic portion of the strain tensor in the plastic zone is determined from Eqs.(2), (8) and (23) as

$$\varepsilon_r^e = \frac{2k}{\sqrt{3}}\left[\sin\left(\psi+\frac{\pi}{3}\right)+\nu\sin\left(\psi-\frac{\pi}{3}\right)\right],$$
$$\varepsilon_\theta^e = -\frac{2k}{\sqrt{3}}\left[\sin\left(\psi-\frac{\pi}{3}\right)+\nu\sin\left(\psi+\frac{\pi}{3}\right)\right], \qquad (27)$$
$$\varepsilon_z^e = -2\nu k\cos\psi.$$

Assume that $p \equiv \psi_b$. Then, differentiating Eq.(27) with respect to ψ_b yields

$$\zeta_r^e = \frac{2k}{\sqrt{3}}\left[\cos\left(\psi+\frac{\pi}{3}\right)+\nu\cos\left(\psi-\frac{\pi}{3}\right)\right]\frac{\partial\psi}{\partial\psi_b},$$
$$\zeta_\theta^e = -\frac{2k}{\sqrt{3}}\left[\cos\left(\psi-\frac{\pi}{3}\right)+\nu\cos\left(\psi+\frac{\pi}{3}\right)\right]\frac{\partial\psi}{\partial\psi_b}, \qquad (28)$$
$$\zeta_z^e = 2\nu k\sin\psi\,\frac{\partial\psi}{\partial\psi_b}.$$

The derivative $\partial\psi/\partial\psi_b$ can be found from Eq.(25). In particular, differentiating ρ^2 gives

$$\frac{2\rho}{b^2}d\rho = \exp\left[\sqrt{3}\left(\psi-\psi_b\right)\right]\left(\sqrt{3}-\cot\psi\right)\frac{\sin\psi_b}{\sin\psi}d\psi$$
$$+\exp\left[\sqrt{3}\left(\psi-\psi_b\right)\right]\frac{\left(\cos\psi_b-\sqrt{3}\sin\psi_b\right)}{\sin\psi}d\psi_b. \qquad (29)$$

Eliminating ρ in this equation by means of Eq.(25) results in

$$\frac{\partial\psi}{\partial\psi_b} = \frac{\cos\left(\psi_b+\pi/3\right)}{\cos\left(\psi+\pi/3\right)}\frac{\sin\psi}{\sin\psi_b}. \qquad (30)$$

Substituting Eq.(30) into Eq.(28) yields

$$\zeta_r^e = \frac{2k}{\sqrt{3}}\left[\cos\left(\psi+\frac{\pi}{3}\right)+\nu\cos\left(\psi-\frac{\pi}{3}\right)\right]\frac{\cos\left(\psi_b+\pi/3\right)}{\cos\left(\psi+\pi/3\right)}\frac{\sin\psi}{\sin\psi_b},$$
$$\zeta_\theta^e = -\frac{2k}{\sqrt{3}}\left[\cos\left(\psi-\frac{\pi}{3}\right)+\nu\cos\left(\psi+\frac{\pi}{3}\right)\right]\frac{\cos\left(\psi_b+\pi/3\right)}{\cos\left(\psi+\pi/3\right)}\frac{\sin\psi}{\sin\psi_b}, \qquad (31)$$
$$\zeta_z^e = 2\nu k\frac{\cos\left(\psi_b+\pi/3\right)}{\cos\left(\psi+\pi/3\right)}\frac{\sin^2\psi}{\sin\psi_b}.$$

It is evident from Eq.(3) that the thermal portions of the total strain rates are independent of ρ. Therefore, the equation of strain rate compatibility is equivalent to

$$\rho \frac{\partial\left(\zeta_\theta^e + \zeta_\theta^p\right)}{\partial \rho} = \zeta_r^e + \zeta_r^p - \zeta_\theta^e - \zeta_\theta^p . \tag{32}$$

Substituting Eq.(23) into Eq.(7) and eliminating λ between the first two equations lead to

$$\zeta_r^p - \zeta_\theta^p = \frac{\sqrt{3}\zeta_\theta^p \sin\psi}{\cos\left(\psi + \pi/3\right)} . \tag{33}$$

Replacing differentiating with respect to ρ with differentiating with respect to ψ in Eq.(32) by means of Eq.(24) and using Eq.(33) give

$$\frac{\partial\left(\zeta_\theta^e + \zeta_\theta^p\right)}{\partial \psi} + \sqrt{3}\left(\zeta_\theta^e + \zeta_\theta^p\right) = \sqrt{3}\zeta_\theta^e - \frac{\cos\left(\psi + \pi/3\right)}{\sin\psi}\left(\zeta_r^e - \zeta_\theta^e\right) . \tag{34}$$

Eliminating here ζ_r^e and ζ_θ^e by means of Eq.(31) gives

$$\frac{\partial\left(\zeta_\theta^e + \zeta_\theta^p\right)}{\partial \psi} + \sqrt{3}\left(\zeta_\theta^e + \zeta_\theta^p\right) = -\frac{k\sin\left(\psi_b - \pi/6\right)}{\sqrt{3}\sin\psi_b \sin\left(\psi - \pi/6\right)}\left[2 - \nu - \left(1 - 2\nu\right)\cos 2\psi\right] . \tag{35}$$

The general solution to this linear differential equation for $\zeta_\theta^e + \zeta_\theta^p$ is

$$\begin{gathered} \zeta_\theta^p + \zeta_\theta^e = k\exp\left(-\sqrt{3}\psi\right)\left[\zeta_0 - \frac{\sin\left(\psi_b - \pi/6\right)}{\sqrt{3}\sin\psi_b}\int\limits_{\psi_c}^{\psi} \Lambda(\chi)\exp\left(\sqrt{3}\chi\right)d\chi\right], \\ \Lambda(\chi) = \frac{2 - \nu - \left(1 - 2\nu\right)\cos 2\psi}{\sin\left(\psi - \pi/6\right)} . \end{gathered} \tag{36}$$

Here ζ_0 is a constant of integration and χ is a dummy variable of integration. The solution given by Eqs.(13) and (14) is valid in the range $\rho_c \le \rho \le 1$. However, *A* and *B* are not determined from Eqs.(17) – (19). Nevertheless, Eq.(18) is valid. It follows from Eqs.(14) that

$$\zeta_\theta^e = k\left[\left(1 - \nu\right)\frac{dB}{d\psi_b} - \frac{\left(1 + \nu\right)}{\rho^2}\frac{dA}{d\psi_b}\right] \tag{37}$$

in the range $\rho_c \le \rho \le 1$. The radial stress must be continuous across the elastic/plastic boundary. The material just on the elastic side of the elastic/plastic boundary must satisfy the yield criterion (Hill, 1950). Therefore, the circumferential stress is also continuous across the elastic/plastic boundary. Using Eqs.(14) and (23) these two conditions are represented as

$$\begin{aligned}\frac{2}{\sqrt{3}}\sin\left(\psi_c+\frac{\pi}{3}\right)&=\frac{A}{b^2}\frac{\sin\psi_c}{\sin\psi_b}\exp\left[\sqrt{3}\left(\psi_b-\psi_c\right)\right]+B,\\ \frac{2}{\sqrt{3}}\sin\left(\psi_c-\frac{\pi}{3}\right)&=\frac{A}{b^2}\frac{\sin\psi_c}{\sin\psi_b}\exp\left[\sqrt{3}\left(\psi_b-\psi_c\right)\right]-B.\end{aligned} \tag{38}$$

Here ρ_c has been eliminated by means of Eq.(26). Solving Eq.(38) for *A* and *B* yields

$$\frac{A}{b^2}=\frac{\sin\psi_b}{\sqrt{3}}\exp\left[\sqrt{3}\left(\psi_c-\psi_b\right)\right],\quad B=\cos\psi_c\,. \tag{39}$$

Let ζ_c^e be the value of ζ_θ^e on the elastic side of the elastic/plastic boundary $\rho=\rho_c$. Then, it follows from Eqs.(26), (37) and (39) that

$$\zeta_c^e=2k\sin\psi_c\left[\frac{(1+\nu)}{\sqrt{3}}\frac{\sin\left(\psi_b-\pi/6\right)}{\sin\psi_b}-\frac{d\psi_c}{d\psi_b}\right]. \tag{40}$$

Let ζ_c^p be the value of $\zeta_\theta^e+\zeta_\theta^p$ on the plastic side of the elastic/plastic boundary $\rho=\rho_c$. It follows from Eq.(36) that

$$\zeta_c^p=k\zeta_0\exp\left(-\sqrt{3}\psi_c\right). \tag{41}$$

Since ζ_θ must be continuous across the elastic/plastic boundary, it is evident that $\zeta_c^p=\zeta_c^e$. Therefore, it follows from Eqs.(40) and (41) that

$$\zeta_0=2\exp\left(\sqrt{3}\psi_c\right)\sin\psi_c\left[\frac{(1+\nu)}{\sqrt{3}}\frac{\sin\left(\psi_b-\pi/6\right)}{\sin\psi_b}-\frac{d\psi_c}{d\psi_b}\right] \tag{42}$$

The solution given by Eqs.(15) and (16) is valid in the range $\rho_c \le \rho \le 1$. However, A_H and B_H are not determined from Eqs.(17) – (19). Nevertheless, Eq.(17) is valid. Therefore, it follows from Eqs.(8), (15) and (16) that

$$\frac{\sigma_r}{\sigma_H}=B_H\left(1-\frac{a^2}{\rho^2}\right),\qquad \varepsilon_\theta=k_H B_H\left[1-\nu+(1+\nu)\frac{a^2}{\rho^2}\right]+k\tau\,. \tag{43}$$

Substituting Eqs.(23) and (43) into Eq.(12)[1] and using Eq.(8) give

$$B_H = \frac{2\gamma b^2}{\sqrt{3}\left(b^2 - a^2\right)} \sin\left(\psi_b + \frac{\pi}{3}\right). \tag{44}$$

A consequence of Eq.(12)[2] is $[\zeta_\theta] = 0$ at $\rho = b$. Then, it follows from Eqs.(8), (36) and (43) that

$$\exp\left(-\sqrt{3}\psi_b\right)\left[\zeta_0 - \frac{\sin\left(\psi_b - \pi/6\right)}{\sqrt{3}\sin\psi_b}\int_{\psi_c}^{\psi_b} \Lambda(\chi)\exp\left(\sqrt{3}\chi\right)d\chi\right] = \frac{dB_H}{d\psi_b}\left[1 - \nu + (1+\nu)\frac{a^2}{b^2}\right]\frac{\beta}{\gamma}. \tag{45}$$

Differentiating Eq.(44) with respect to ψ_b leads to

$$\frac{dB_H}{d\psi_b} = \frac{2\gamma b^2}{\sqrt{3}\left(b^2 - a^2\right)} \cos\left(\psi_b + \frac{\pi}{3}\right). \tag{46}$$

Eliminating $dB_H/d\psi_b$ and ζ_0 in Eq.(45) by means of Eqs.(46) and (42) gives

$$\begin{aligned} \frac{d\psi_c}{d\psi_b} = {} & \frac{(1+\nu)}{\sqrt{3}}\frac{\sin\left(\psi_b - \pi/6\right)}{\sin\psi_b} - \frac{\sin\left(\psi_b - \pi/6\right)}{2\sqrt{3}\sin\psi_b\sin\psi_c}\int_{\psi_c}^{\psi_b}\Lambda(\chi)\exp\left[\sqrt{3}\left(\chi - \psi_c\right)\right]d\chi - \\ & - \frac{\beta b^2}{\sqrt{3}\left(b^2 - a^2\right)}\frac{\exp\left[\sqrt{3}\left(\psi_b - \psi_c\right)\right]}{\sin\psi_c}\cos\left(\psi_b + \frac{\pi}{3}\right)\left[1 - \nu + (1+\nu)\frac{a^2}{b^2}\right]. \end{aligned} \tag{47}$$

The boundary condition to this equation is

$$\psi_b = \psi_c = \psi_e. \tag{48}$$

Here ψ_e is the value of ψ at $\rho = b$ at the instant of the initiation of plastic yielding. Therefore, it follows from Eqs.(13) and (23) that

$$\frac{A}{b^2} + B = \frac{2}{\sqrt{3}}\sin\left(\psi_e + \frac{\pi}{3}\right), \quad \frac{A}{b^2} - B = \frac{2}{\sqrt{3}}\sin\left(\psi_e - \frac{\pi}{3}\right). \tag{49}$$

In this equation *A* and *B* are determined from Eqs.(17) – (19) at $\tau = \tau_b$. It is evident that a unique value of ψ_e in the range $0 \le \psi_e < 2\pi$ is determined from Eq.(49). Then, Eq.(47) should be solved numerically using the boundary condition in Eq.(48). Once the solution to Eq.(47) has been found the distribution of stress is determined from Eq.(15)

in the range $a \le \rho \le b$, from Eqs.(23) and (25) in the range $b \le \rho \le \rho_c$ and from Eq.(13) in the range $\rho_c \le \rho \le 1$. The values of A_H and B_H are found as functions of ψ_b from Eqs.(17) and (44). The values of A and B are found as functions of ψ_b from Eq.(39) and the solution to Eq.(47). Thus the distribution of stress depends on ρ and ψ_b. The latter can be replaced with τ by means of Eq.(18). There are two restrictions on the solution found. One of these restrictions can be derived from Eq.(22)^2 in which A_H and B_H should be eliminated using the elastic/plastic solution. The right hand side of this equation must be less or equal to 1. The other restriction is $\rho_c \le 1$. Using Eq.(26) this restriction is represented as

$$b\sqrt{\frac{\sin\psi_b}{\sin\psi_c}}\exp\left[\frac{\sqrt{3}}{2}\left(\psi_c - \psi_b\right)\right] \le 1. \quad (50)$$

5. DESTRIBUTION OF RESIDUAL STRESSES

On release of τ an elastic recovery precedes any possible reversed plastic yielding. The question as to whether reversed plasticity occurs depends upon the magnitude of τ at the end of loading. The solution given in this section is restricted to this range of τ corresponding to a purely elastic recovery. In this case, the general solutions described by Eqs.(13) – (16) are valid for the increments of stress and strain. Therefore,

$$\frac{\Delta\sigma_r}{\sigma_0} = \frac{\Delta A}{\rho^2} + \Delta B, \qquad \frac{\Delta\sigma_\theta}{\sigma_0} = -\frac{\Delta A}{\rho^2} + \Delta B, \quad \Delta\varepsilon_\theta = k\left[\left(1-\nu\right)\Delta B - \left(1+\nu\right)\frac{\Delta A}{\rho^2} - \tau_m\right] \quad (51)$$

in the range $b \le \rho \le 1$ and

$$\frac{\Delta\sigma_r}{\sigma_H} = \frac{\Delta A_H}{\rho^2} + \Delta B_H, \qquad \frac{\Delta\sigma_\theta}{\sigma_H} = -\frac{\Delta A_H}{\rho^2} + \Delta B_H, \quad \Delta\varepsilon_\theta = k_H\left[\left(1-\nu\right)\Delta B_H - \left(1+\nu\right)\frac{\Delta A_H}{\rho^2}\right] - k\tau_m \quad (52)$$

in the range $a \le \rho \le b$. In these equations, ΔA, ΔB, ΔA_H, and ΔB_H are new constants of integration and τ_m is the value of τ at the end of loading. The boundary conditions shown in Eq.(10) and (11) transform to

$$\Delta\sigma_r = 0 \quad (53)$$

for $\rho = a$ and

$$\Delta\varepsilon_\theta = 0 \quad (54)$$

for $\rho = 1$. Combining Eqs.(51) – (54) gives

$$\Delta A = \frac{(1-\nu)\Delta B - \tau_m}{1+\nu}, \quad \Delta A_H = -a^2 \Delta B_H . \tag{55}$$

Equations (12) become $[\Delta\sigma_r] = 0$ and $[\Delta\varepsilon_\theta] = 0$ at $\rho = b$. Then, it follows from Eqs.(8), (51) and (52) that

$$\begin{aligned} &\left(\Delta A + \Delta B b^2\right)\gamma = \Delta A_H + \Delta B_H b^2, \\ &(1-\nu)\Delta B b^2 - (1+\nu)\Delta A = \left[(1-\nu)\Delta B_H b^2 - (1+\nu)\Delta A_H\right]\frac{\beta}{\gamma}. \end{aligned} \tag{56}$$

Solving Eq.(55) and (56) for ΔB yields

$$\begin{aligned} &\Delta B = -\tau_m \frac{C_2}{C_1}, \\ &C_1 = (1-\nu)\left(b^2 - 1\right) - \frac{\beta}{\left(b^2 - a^2\right)}\left[(1-\nu)b^2 + (1+\nu)a^2\right]\left(\frac{1-\nu}{1+\nu} + b^2\right), \\ &C_2 = 1 + \left(\frac{1-\nu}{1+\nu} + \frac{a^2}{b^2}\right)\frac{b^2\beta}{\left(b^2 - a^2\right)}. \end{aligned} \tag{57}$$

Then, ΔA can be found from Eq.(55)[1]. Having ΔA and ΔB it is possible to determine ΔB_H from

$$\Delta B_H = \frac{\left(\Delta A + \Delta B b^2\right)\gamma}{b^2 - a^2}. \tag{58}$$

Finally, ΔA_H is found from Eq. Eq.(55)[2]. Substituting ΔA, ΔB, ΔA_H and ΔB_H into Eqs.(51) and (52) yields the distribution of $\Delta\sigma_r$ and $\Delta\sigma_\theta$. Then, the distribution of residual stresses is found as

$$\sigma_r^{res} = \sigma_r + \Delta\sigma_r, \quad \sigma_\theta^{res} = \sigma_\theta + \Delta\sigma_\theta. \tag{59}$$

Here σ_r and σ_θ are understood at the end of loading. The restrictions of the validity of the solution given in this section are

$$\left(\sigma_r^{res}\right)^2 + \left(\sigma_\theta^{res}\right)^2 - \sigma_\theta^{res}\sigma_r^{res} - \sigma_0^2 \le 0 \tag{60}$$

in the range $b \le \rho \le 1$ and

$$\left(\sigma_r^{res}\right)^2 + \left(\sigma_\theta^{res}\right)^2 - \sigma_\theta^{res}\sigma_r^{res} - \sigma_H^2 \le 0 \qquad (61)$$

in the range $a \le \rho \le b$.

6. ILLUSTRATIVE EXAMPLE

Assume that $\gamma = 1/3.9$ and $\beta = 1/1.7$ (Cho et al, 2012). Consider a disc with $a = 0.3$, $k = 10^{-3}$ and $\nu = 0.3$. Equation (47) has been solved numerically. Then, the radial and circumferential stresses have been found as functions of ρ and ρ_c using formulae derived in Section 4. The variation of the radial and circumferential stresses with ρ at $b = 0.31$ and several values of ρ_c is depicted in Figs. 1 and 2, respectively. For values of $\rho_c > 0.83$ (approximately) plastic yielding starts at $\rho = a$. In order to illustrate the effect of the thickness of the hard layer on the distribution of stresses, the variation of the radial and circumferential stresses with ρ at $\rho_c = 0.8$ and several values of b is depicted in Figs. 3 and 4, respectively. It is seen from Figs. 1 to 4 that the hard layer has a significant effect of the radial stress in the base material and of the circumferential stress with the hard layer. The residual radial and circumferential stresses have been found using formulae derived in Section 5. The variation of the residual radial and circumferential stresses with ρ at $b = 0.31$ and several values of ρ_c is depicted in Figs. 5 and 6, respectively, and at $\rho_c = 0.8$ and several values of b in Figs. 7 and 8. It has been verified that the conditions shown in Eqs.(60) and (61) are satisfied. It is seen from Figs.5 to 8 that the hard layer has a larger effect on the residual circumferential stress as compared to the residual radial stress.

7. CONCLUSIONS

A new semi-analytical plane-stress elastic-plastic solution for a disc with a concentric hole inserted into a container and subject to thermal loading has been derived to determine the distribution of stresses at the end of loading and the distribution of residual stresses after unloading. The numerical treatment of the boundary value problem has been reduced to solving an ordinary differential equation and evaluating ordinary integrals. The solution is restricted to the range of material and process parameters corresponding to purely elastic response on the material within the hard layer and to purely elastic release. It is has been assumed that there is a narrow hard layer in the vicinity of the hole generated by a preceding treatment of the surface. The numerical example has been provided for the actual difference between the yield stress and Young modulus within the layer and base material found in Cho et al (2012). It has been shown that the predicted distribution of the radial stress at the end of loading is influenced by the thickness of the hard layer whereas the distribution of the circumferential stress within the base material is not. On the other hand, the distribution of residual circumferential stress is significantly affected by the presence of the hard layer. These results call for a systematic study on the effect of hard (white) layers on the response of structures under various loading conditions.

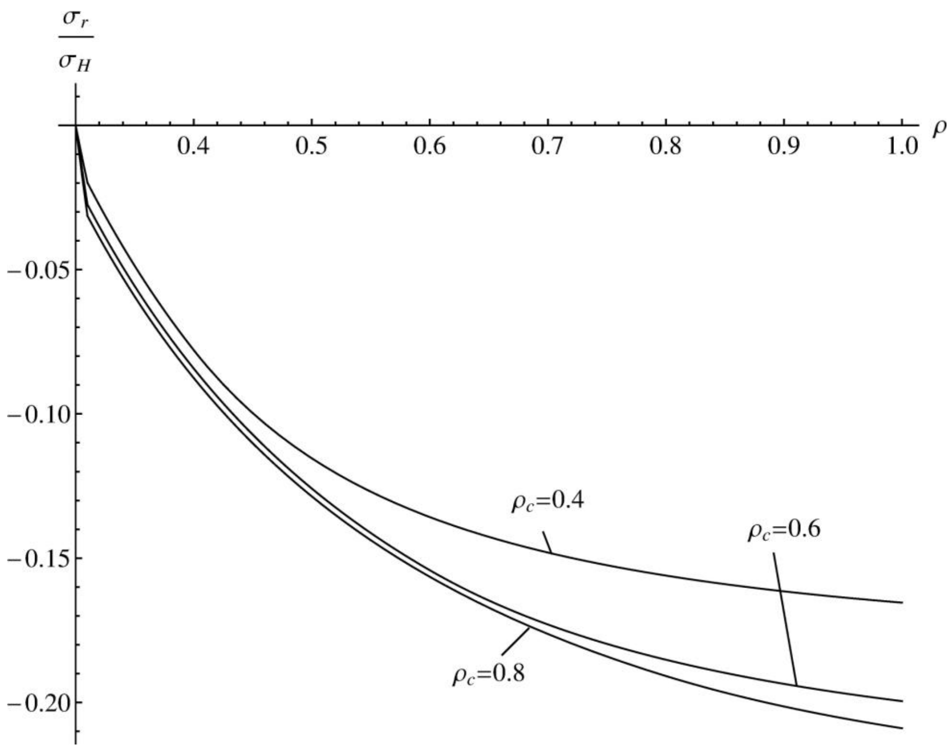

Fig. 1 Variation of the radial stress with ρ at $b = 0.31$ and several values of ρ_c

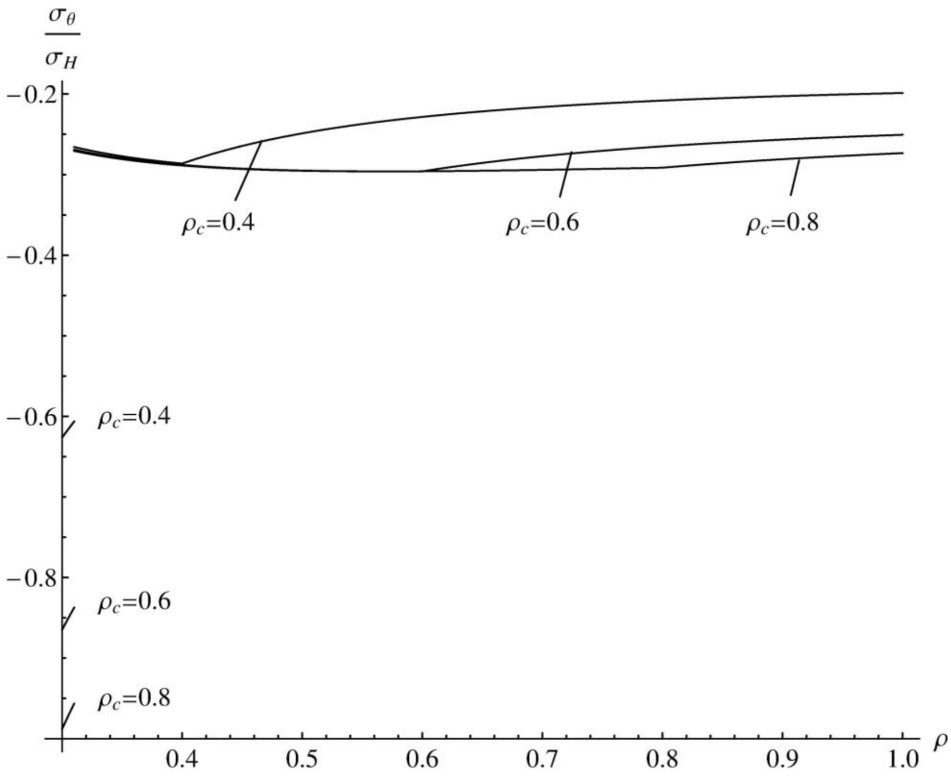

Fig. 2 Variation of the circumferential stress with ρ at $b = 0.31$ and several values of ρ_c

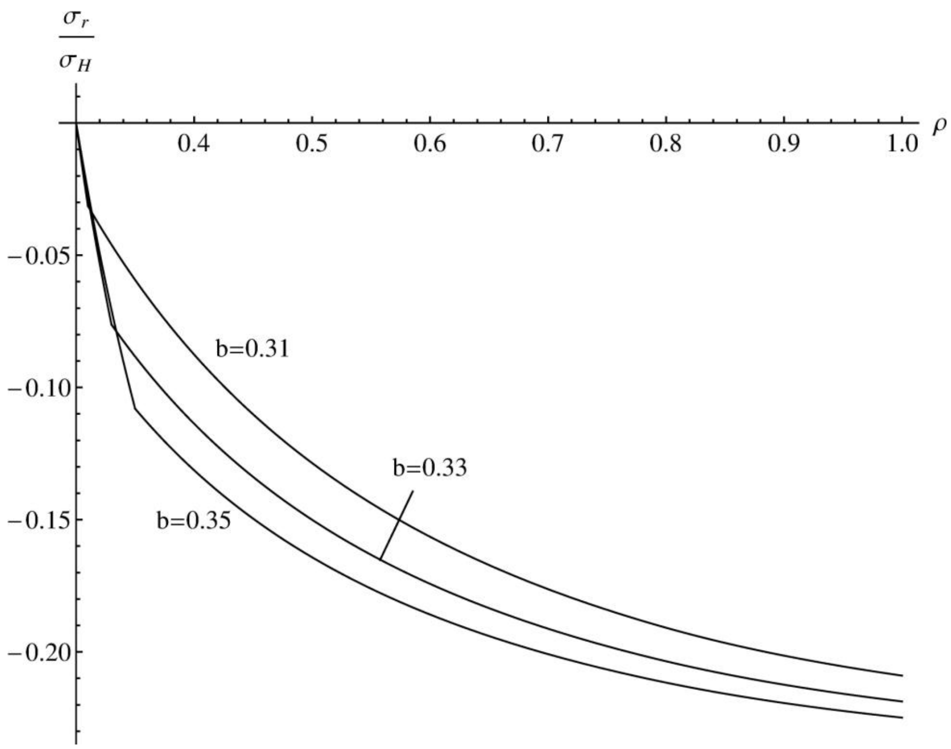

Fig. 3 Variation of the radial stress with ρ at $\rho_c = 0.8$ and several values of *b*

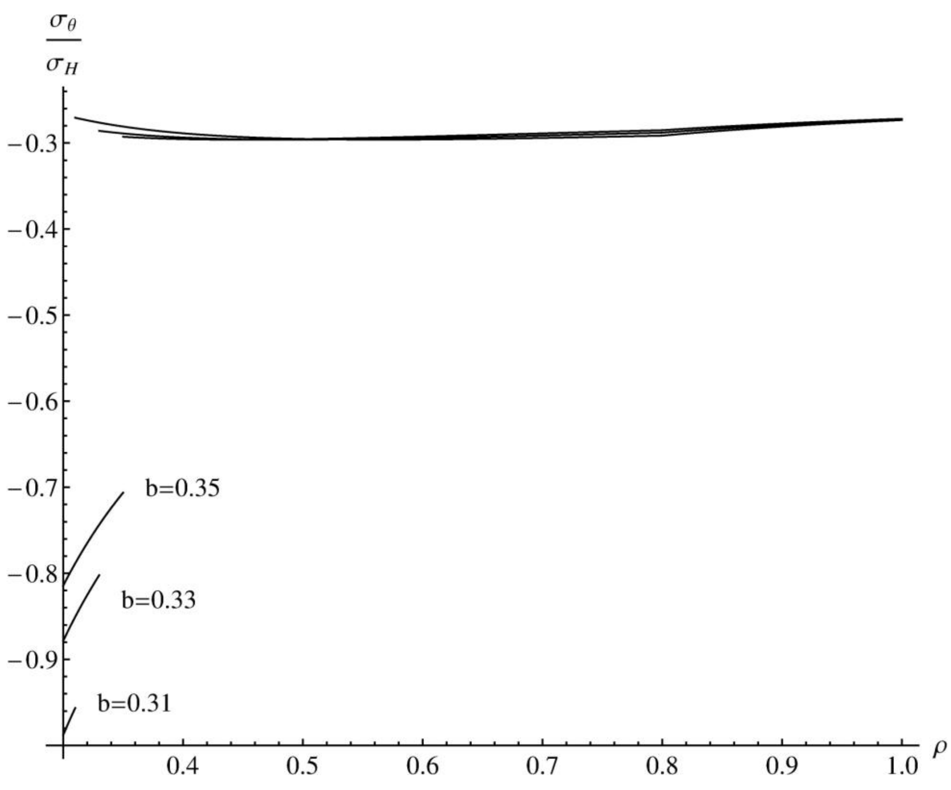

Fig. 4 Variation of the circumferential stress with ρ at $\rho_c = 0.8$ and several values of *b*

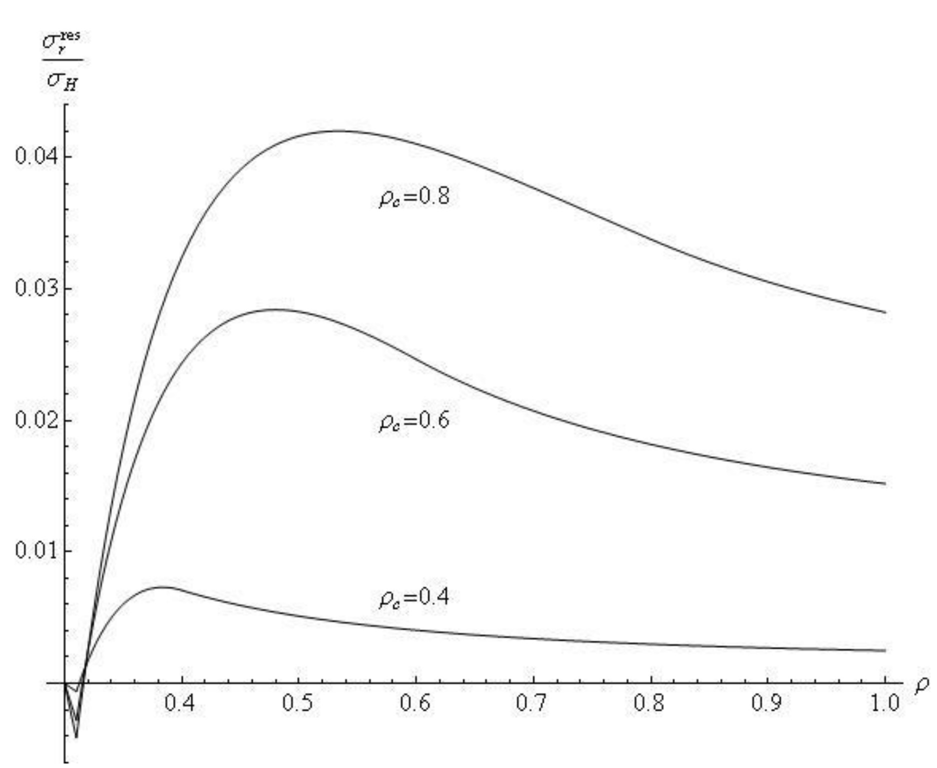

Fig. 5 Variation of the residual radial stress with ρ at $b = 0.31$ and several values of ρ_c

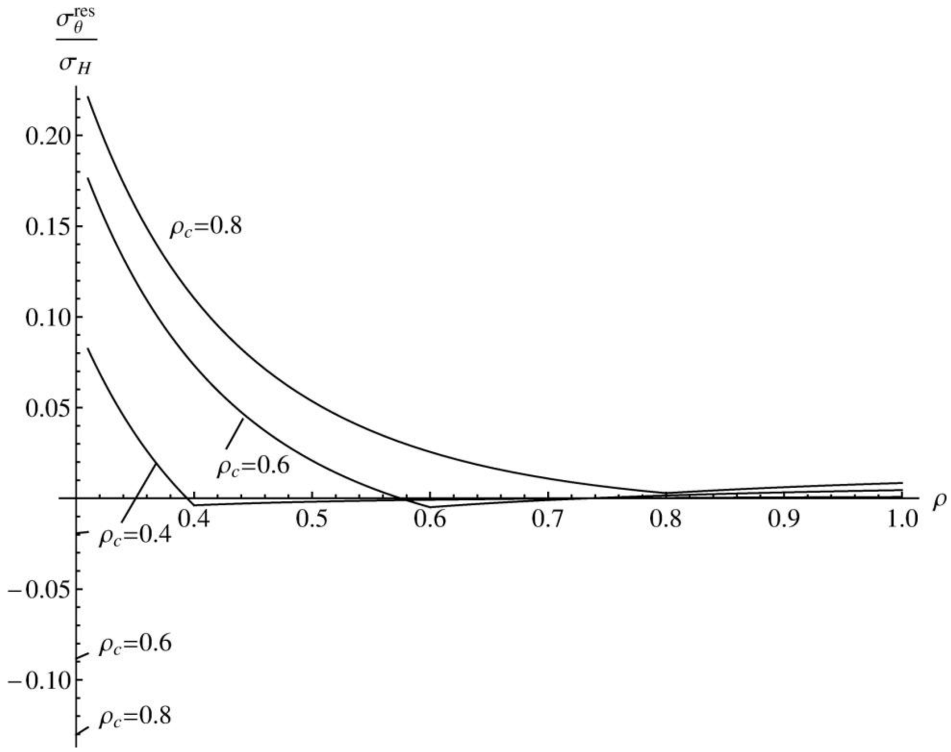

Fig. 6 Variation of the residual circumferential stress with ρ at $b = 0.31$ and several values of ρ_c

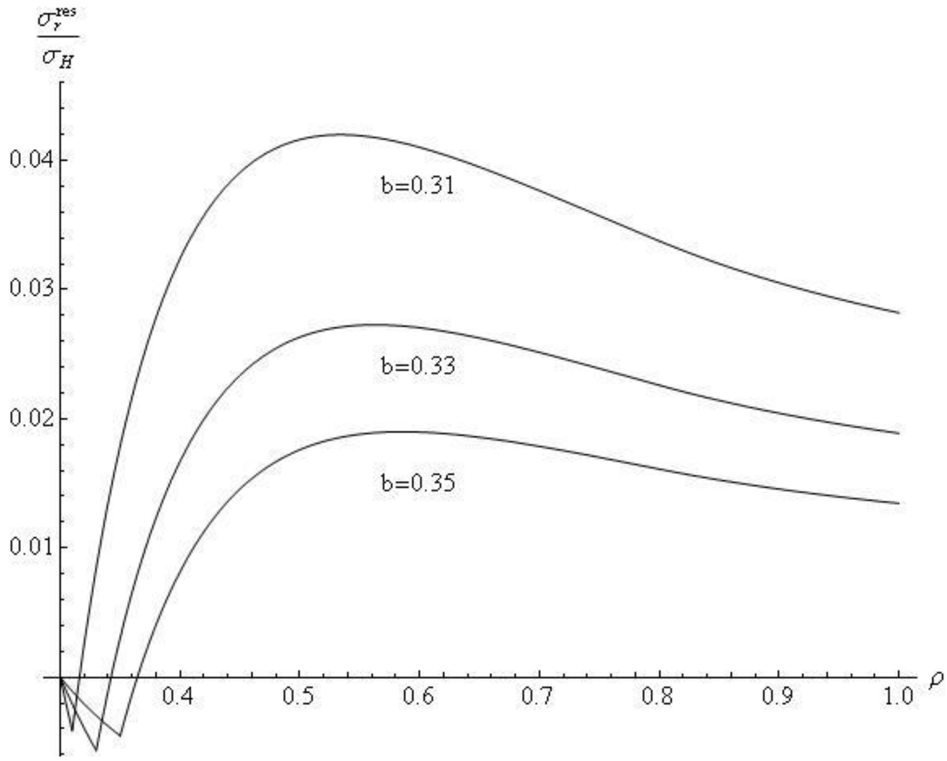

Fig. 7 Variation of the residual radial stress with ρ at $\rho_c = 0.8$ and several values of ***b***

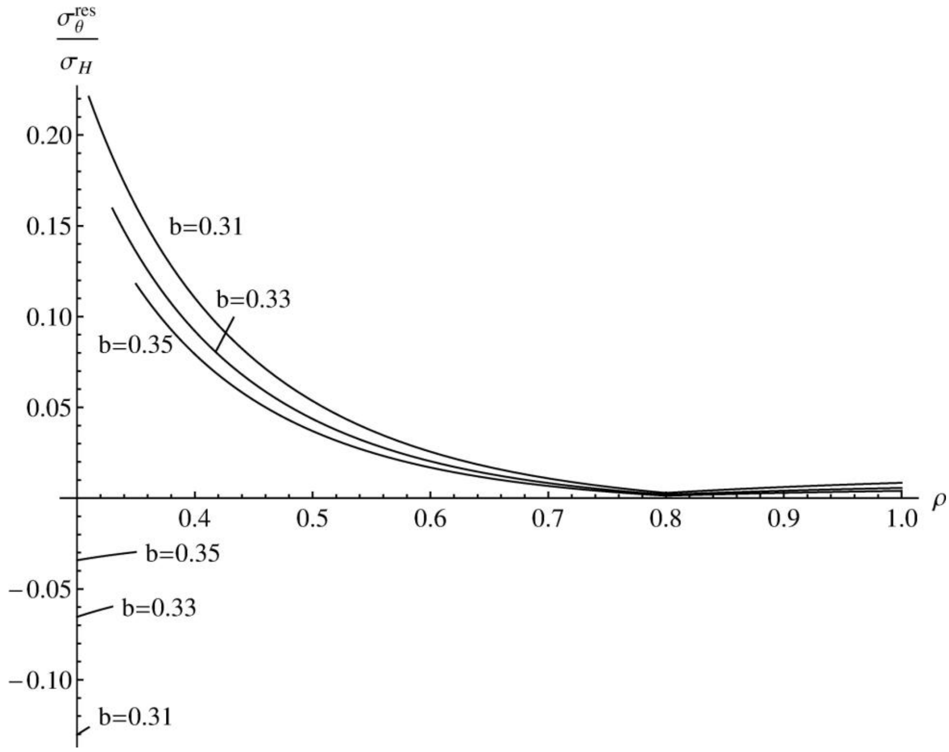

Fig. 8 Variation of the residual circumferential stress with ρ at $\rho_c = 0.8$ and several values of ***b***

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