$$M_c = \frac{u_1 - u_2}{c_1 - c_2} = 0.6 \tag{27}$$

An oblique shock with a shock-angle of $\beta = 12°$ is imposed on the upper boundary, and a slip wall condition is applied to the lower boundary. Periodic boundary condition is applied for both ends of z-surfaces. Fluctuation adding to the mean in-flow is given by

$$v' = \sum_{k=1}^{2} a_k \cos(2\pi k\, t/T + z/L_z + \phi_k) \exp(-y^2/b), \tag{28}$$

with a period $T = \lambda/u_c$, a wavelength $\lambda = 30$ and a convective velocity $u_c = 2.68$. The other parameters are as follows: $a_1 = a_2 = 0.05$, $\phi_1 = 0$, $\phi_2 = \pi/2$ and $b = 10$. L_z, the extrusion length, is 40. The Reynolds number and the Prandtl number are 500 and 0.72, respectively. The computational domain is $[0, 200] \times [-20, 20] \times [-20, 20]$. Grid system consists of 3.5 million tetrahedral elements. With Tachyon 2 supercomputer at KISTI, MPI parallel computation was performed with 512 CPUs to reach at $t = 120$. For a better resolution, many filter methods have computed this problem on meshes clustered along the *y*-direction, but the present computation employs uniformly distributed triangular grids of $h = 0.75$.

Figure 3 shows the density contour and iso-surfaces at $t = 120$. Due to the three-dimensional perturbation (Eq. (28)), phase difference is induced along the *z*-direction. Before the oblique shock strikes the mixing layer, spanwise vortical structure is regularly developed along the *z*-direction, and after the first oblique shock-mixing layer interaction, spanwise vortical structure is noticeably deformed. After the reflected shock hits the mixing layer again, spanwise shock-vortex interaction is further developed. Higher-order approximation/reconstruction with MLP maintains the vortical structure along the downstream field.

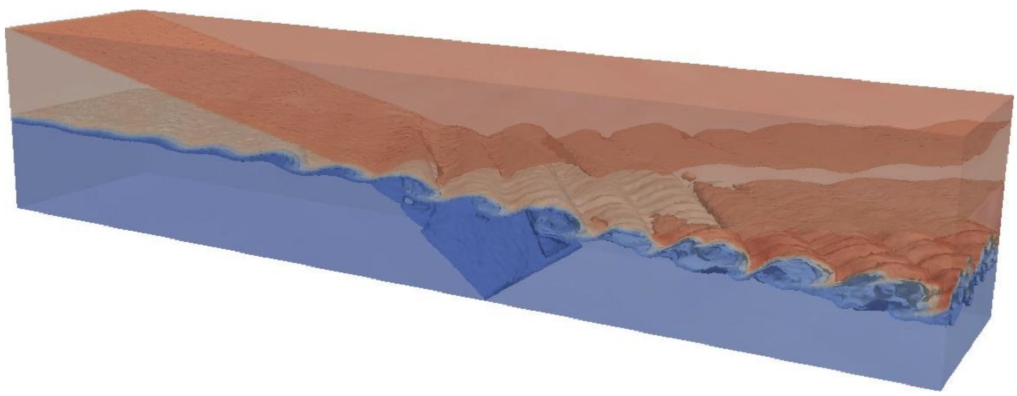

Figure 3: Density contours of three dimensional oblique shock-mixing layer interaction at $t = 120$.

4. ISSUES FOR FURTHER IMPROVEMENTS

Extensive numerical experiments validate the robustness and accuracy of the hierarchical MLP with higher-order methods in compressible inviscid and viscous flow simulations. At the same time, there are further rooms to improve and extend this limiting philosophy to deal with high-Reynolds number compressible flows around realistic configurations. Several issues are selected for discussion as follows:

4.1 Sub-cell Resolution across Shock Waves

As other limiter-based approaches, MLP enforces the monotonicity of cell-averaged values only. Thus, it may not completely control the sub-cell resolution around a shock wave. As observed in previous researches (Park submitted), current hierarchical MLP yields slight overshoots in the sub-cell distribution of a numerical shock. This may grow as a potential source of numerical instability if the order-of-accuracy is further increased, because the MLP condition is only imposed on the averaged value of the cell in which there are many degrees of freedom. Recently, there are some studies combining *hp*-refinement and limiters (Dumbser 2014). This approach requires both higher-order DG/CPR and FVM solvers, and computational overheads may become serious while switching between these solvers. Unlike limiters, artificial viscosity-based approaches does not seem to suffer from this issue. Instead, there are tuning parameters to determine the diffusion, which usually relies on the flow structure, grid system and the desired order-of-accuracy (Persson 2006). Despite some progresses to determine such tuning parameters, it appears that artificial viscosity is not robust enough yet to resolve very strong shock and expansion waves (Yu 2014, Park 2014 (ICCFD)). In the context of MLP concept, more research efforts needs to be directed to this issue by controlling the limited approximation or reconstruction.

4.2 Convergence for Steady Flow Problems

Up to now, the hierarchical MLP has been developed mainly for unsteady flow problems. While its performances are validated, convergence issue for steady-state problems still remains. It is well-known that slope limiters whose operations are non-differentiable, may fail to reach a convergent solution even in finite volume methods. Such non-differentiable operators are sensitive to the numerical fluctuations near shock wave and they may become stalled. While we proposed the MLP-u2 slope limiters to overcome this issue (Park 2010, Park 2012), the troubled-cell detecting mechanism and projection operator for the DG and CPR methods are still non-differentiable. In addition, higher-order CFD methods have a reduced damping mechanism for transient error, thus this situation may become more problematic. Implicit time integrations are preferred to compute steady flow problems, but it is quite untractable to construct an implicit operator for non-differentiable hierarchical MLP. It appears that additional smooth transition mechanism between troubled-cell and normal-cell is necessary.

4.3 Interaction of Turbulent Vortices with Shock Waves

One of the promising areas for higher-order methods is turbulent flows. Especially, researchers expect higher-order schemes to accurately resolve interactions between shock and turbulent vortices around high-speed vehicles. Recent researches attempt to calculate turbulence by Reynolds-averaged Navier-Stokes (RANS) equations with Spalart-Allmaras model (Nguyen 2007) and $k-\omega$ model (Bassi 2005), and by DNS/LES approaches (Wang 2013). Numerical experiments reveal that current higher-order CFD methods are not robust as finite volume RANS solvers, primarily due to the nonsmoothness introduced in turbulence models (Wang 2013). Recently, there are some progresses to improve the accuracy of turbulence models by modifying closure models (Nguyen 2007, Allmaras 2012), by developing hybrid RANS-LES models (Spalart 1997, Spalart 2006), or by introducing transition models (Langtry 2009). The enhanced resolution of higher-order methods makes it possible to accurately simulate highly unsteady turbulent flow and/or laminar-turbulent transitions with improved turbulence and/or transition models. However, there are few studies to simulate turbulent flows with shock waves with higher-order methods. Since excessive numerical viscosity of shock-capturing schemes may easily dissipate small scale vortical structure, accurate shock-capturing schemes are indeed essential. From successful numerical experiments, we are expecting that the hierarchical MLP limiting may provide a proper dissipation-control mechanism to capture detailed turbulent flows as well as shock waves.

5. CONCLUSIONS

Guided by the MLP condition and the maximum principle, the hierarchical MLP limiting is successfully extended into the higher-order CFD methods such as the DG and CPR methods. The extended forms of the MLP condition, *i.e.*, the augmented MLP condition and the $P1$-projected MLP condition, are proposed to treat the solution points near discontinuities without compromising the higher-order nature in smooth region. The uncertainty of determining a parameter for slope limiting is then eliminated by examining the behavior of local extrema near vertex point. Finally, the hierarchical MLP limiting is formulated by combining one of the extended MLP conditions with the MLP extrema detector.

Extensive computations, ranging from scalar conservation laws to multi-dimensional flow systems, are carried out up to *P3* approximation to examine the capability of the hierarchical MLP methods in capturing multi-dimensional flow physics. Numerous comparisons and grid refinement tests on unstructured grids demonstrate the proposed limiting provides detailed multi-dimensional flow structures without numerical oscillations in discontinuous region, while maintaining the required accuracy in smooth region. The hierarchical MLP limiting is robust and efficient in the sense that it does not require any tuning parameter and it is applied to conservative variables without characteristic decomposition. At the same time, more efforts need to be exerted in the areas of sub-cell resolution, steady-state convergence and turbulence/transition models in order to extend the current approach to the simulations of high-Reynolds number compressible flows around realistic configurations.

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