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The minimum and maximum elastic buckling strengths of unbraced frames subjected to variable Loading

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ABSTRACT

A fundamental task in structural stability analysis is to ensure the safety of structures throughout their operational life so as to prevent catastrophic consequences triggered by both normal and abnormal loadings. In contrast to current frame stability analysis involving only proportional loads, the variable-loading stability analysis, characterized by a pair of minimization and maximization problems, permits individual applied loads on the frame to vary independently so as to capture the load patterns that cause elastic instability of frames at minimum and maximum load levels. The analysis clearly identifies the stability capacities of unbraced frames under extreme load cases, which is the information that is generally not available through conventional proportional loading stability analysis.

Presented in this paper is a new method to assess the elastic stability of unbraced frames subjected to variable loading featured with the lateral stiffness of an axially loaded column derived by using two cubic Hermite elements to signify the column. Unlike the column stiffness obtained from Euler-Bernoulli beam theory, which contains transcendental functions resulted in difficulties of applying nonlinear programming algorithms to solve the minimization and maximization problems, the stiffness in the proposed method includes only polynomials. Thus, the column stiffness in the proposed method enables the minimization and maximization problems being solved by efficient gradient-based nonlinear programming algorithms. The accuracy on the column stiffness associated with the proposed method is compared with that of Euler-Bernoulli beam theory. Four unbraced steel frames are investigated to demonstrate the efficiency of the proposed method.

1. INTRODUCTION

[2)] Research Assistance
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Structural stability is one of the primary concerns in designing unbraced frames. In the past few decades, extensive research has been conducted on the stability of unbraced frames. Among them, the effective length based methods are widely used in research and design practice. Among those methods, the alignment-chart-based method is most commonly used to evaluate the column buckling strength in engineering practice (AISC 2010). However, it is well known that simplifications and assumptions adopted in the alignment chart-based method may result in inaccurate column buckling strength if assumptions are not satisfied.

Unlike the alignment-chart-based effective length method, which neglects the interactions among columns in the same storey in resisting lateral sway buckling of unbraced frames, the storey-based buckling method introduced by Yura (1971) is based on the fact that lateral sway instability of an unbraced frame is a storey phenomenon involving the interaction of the lateral sway resistance of each column in the same storey and the total gravity load in the columns in that storey. Based on this concept, different methods were proposed to evaluate the buckling strength of unbraced frames (Xu and Wang 2008, Tong et al. 2009, Hellesland 2009, Aristizabal-Ochoa 2011). It must be pointed out that the foregoing researches are almost exclusively based on the assumption of proportional loading, in which predefined load patterns are assigned to the frame. Accordingly, the possible worst load pattern may not always be included in the specified load combinations due to the unpredictable nature of various types of loads. Thus, the variable loadings, both in magnitudes and locations, are needed to be considered when assessing the stability of structures. To this end, the buckling strength of unbraced frame subjected to variable loading was first investigated by Xu et al. (2001) based on the concept of storey-based buckling and with use of linear programming algorithm. In contrast to conventional frame stability analysis which only accounts for proportional loading, the method proposed by Xu et al. (2001) could capture the frame buckling strengths and their associated load patterns that cause instability of unbraced steel frames at the minimum load levels (the worst load pattern) and the maximum load levels (the most favorable pattern). The method enabled design practitioners to evaluate the stability of unbraced frames in extreme loading cases. Considering that Taylor series approximation of the lateral stiffness of an axially loaded column was adopted in the linear programming based method, Xu (2003) subsequently proposed a nonlinear programming based approach in which the column stiffness was derived directly from the Euler-Bernoulli beam theory without any approximation. The problems to determine the minimum and the maximum load levels (the most favourable pattern) and frame buckling strengths were subsequently solved by a non-gradient-based algorithm due to the complexity of the column stiffness being a transcendental function. More recently, Zhuang and Xu (2013) found that the linear programming based method (Xu et al. 2001) failed to differentiate the difference between the minimum and the maximum frame buckling strengths in the case when the end-fixity factors of all columns in the same storey are identical.

In conventional frame stability analysis, cubic Hermite element was adopted to derive the flexural stiffness matrix of beam-column member accounting for the effect of axial loading on bending stiffness of the member. The stiffness matrix is then used to determine the critical load multiplier of the frame subjected to the proportional loading. In this study, the stiffness matrix will be used to derive the lateral stiffness of an axially loaded column in an unbraced frame. Compared to the lateral stiffness derived based on the classical Euler-Bernoulli beam theory, the proposed lateral stiffness equation is in good accuracy and considerably simpler, which enables the gradient-based algorithm being used to evaluate the minimum and maximum frame buckling strengths of unbraced frames subjected to variable loading. Numerical examples are also presented to demonstrate that the accuracy and efficiency of the proposed method.

2. STOREY-BASED BUCKLING OF UNBRACED FRAME

2.1 Lateral stiffness of an axially loaded column

The conventional column effective length method assumes that columns in unbraced frames to buckle simultaneously in lateral sway mode without accounting for the stiffness interaction among columns in the same storey. The storey-based buckling method takes the consideration of the fact that lateral sway of the weaker or heavily loaded columns may be restrained by the lateral bracing provided by the stronger or lightly loaded columns in the same storey. Therefore, axially loaded columns in unbraced frames may buckle in either one of the two modes: rotational non-sway buckling or lateral sway buckling. For lateral sway buckling, the buckling strength of a column modeled by a cubic Hermite element can be accurately predicted with an error less than 1% but the error associated with using a single cubic Hermite element to evaluate column rotational non-sway buckling can be more than 20% which is not acceptable. However, previous studies showed that the error can be reduced to 2% when two such elements were used to model a column in rotational non-sway buckling (Teh 2001, Kwok and Chan 1991). Thus, two elements will be employed in this study to model the axially loaded column and derive its lateral stiffness such that both rotational non-sway and lateral sway buckling strengths can be accurately predicted. The flexural stiffness matrix of beam-column can be derived by using the energy principle as (Chajes 1974)

$$\boldsymbol{k} = \boldsymbol{k}_e + \boldsymbol{k}_g = \left\{ EI \begin{bmatrix} \frac{12}{l^3} & -\frac{6}{l^2} & -\frac{12}{l^3} & -\frac{6}{l^2} \\ & \frac{4}{l} & \frac{6}{l^2} & \frac{2}{l} \\ & Sym & \frac{12}{l^3} & \frac{6}{l^2} \\ & & & \frac{4}{l} \end{bmatrix} - P \begin{bmatrix} \frac{6}{5l} & -\frac{1}{10} & -\frac{6}{5l} & -\frac{1}{10} \\ & \frac{2l}{15} & \frac{1}{10} & -\frac{l}{30} \\ & Sym & \frac{6}{5l} & \frac{1}{10} \\ & & & \frac{2l}{15} \end{bmatrix} \right\} \quad (1)$$

in which *E* and *I* are the Young's Modulus and moment of initial, respectively. *P* is the axial load of the column, and l is the length of the element. It can be seen that the stiffness matrix $\boldsymbol{k}$ is composed of the elastic stiffness matrix $\boldsymbol{k_e}$ and the geometric stiffness matrix $\boldsymbol{k_g}$. The decrease in flexural stiffness due to the presence of axial compressive force can be observed through the negative diagonals of the matrix $\boldsymbol{k_g}$.

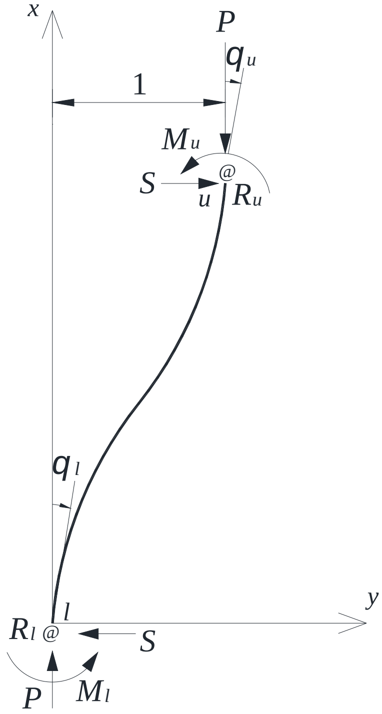

Fig. 1: Axially loaded column

As shown in Fig. 1, the lateral stiffness *S* of an axially loaded column in a frame can be evaluated by imposing a unit lateral deflection at the upper end of the column. The column is evenly subdivided into two elements as plotted in Fig. 2. In Fig. 2, Δ is the nodal displacement of the column, and *W* is the nodal force applied at the column. l_1 and l_2 are the lengths of each element, which equals to half of the length of the column.

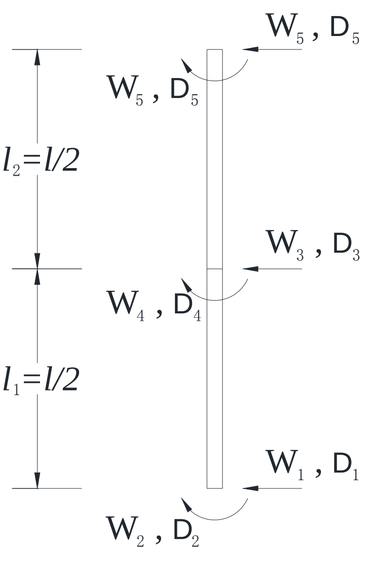

Fig. 2: Nodal forces and displacements for a column

Therefore, based on the stiffness matrix of one element in Eq. (1), the stiffness matrix ***K*** for the column shown in Fig. 2 can be obtained by superimposing the matrices of two elements as follow:

$$\boldsymbol{K}=\left\{EI\begin{bmatrix}\frac{12}{l^3} & -\frac{6}{l^2} & -\frac{12}{l^3} & -\frac{6}{l^2} & 0 & 0\\ & \frac{4}{l} & \frac{6}{l^2} & \frac{2}{l} & 0 & 0\\ & & \frac{24}{l^3} & 0 & -\frac{12}{l^3} & -\frac{6}{l^2}\\ & & & \frac{8}{l} & \frac{6}{l^2} & \frac{2}{l}\\ & Sym & & & \frac{12}{l^3} & \frac{6}{l^2}\\ & & & & & \frac{4}{l}\end{bmatrix}-P\begin{bmatrix}\frac{6}{5l} & -\frac{1}{10} & -\frac{6}{5l} & -\frac{1}{10} & 0 & 0\\ & \frac{2l}{15} & \frac{1}{10} & -\frac{l}{30} & 0 & 0\\ & & \frac{12}{5l} & 0 & -\frac{6}{5l} & -\frac{1}{10}\\ & & & \frac{4l}{15} & \frac{1}{10} & -\frac{l}{30}\\ & Sym & & & \frac{6}{5l} & \frac{1}{10}\\ & & & & & \frac{2l}{15}\end{bmatrix}\right\} \tag{2}$$

Since there is no external force at the mid-point of the column, the nodal forces can be stated as ***W***=[–*S* M_l 0 0 *S* M_u]T. As shown in Fig.1, R_l and R_u are the end rotational restraining stiffness provided by the adjoining members at lower and upper ends of the column, respectively; the corresponding end moments can be therefore calculated as

$$M_u=-R_u\theta_u \tag{3a}$$
$$M_l=-R_l\theta_l \tag{3b}$$

The nodal displacements of the column in Fig. 1 and Fig. 2 are $\Delta=[1\ \theta_u\ \Delta_3\ \Delta_4\ 0\ \theta_l]^T$, in which the displacements at mid-point of the column are unknown and noted as Δ_3 and Δ_4 . The force-displacement relationship for the column in Fig. 1 and Fig. 2 can be stated as

$$\boldsymbol{W}=\boldsymbol{K\Delta} \tag{4}$$

in which ***K*** is the stiffness matrix of the column defined in Eq. (2). Since there is no translational displacement at the lower end, the 6×6 stiffness matrix ***K*** can be reduced to a 5×5 matrix. Substituting Eq. (2), Eqs. (3), ***W*** and Δ into Eq. (4), five equations can be obtained as

$$-\frac{R_l\theta_l}{l_1}=\frac{EI}{l_1^3}\left(-6\Delta_3+2\Delta_4 l_1+4\theta_l l_1\right)-\frac{P}{l_1}\left(-\frac{\Delta_3}{10}-\frac{\Delta_4 l_1}{30}+\frac{2\theta_l l_1}{15}\right) \tag{5}$$

$$0=\frac{EI}{l_1^3}\left(-12+6\theta_u l_1+24\Delta_3-6\theta_l l_1\right)-\frac{P}{l_1}\left(-\frac{6}{5}+\frac{\theta_u l_1}{10}+\frac{12\Delta_3}{5}-\frac{\theta_l l_1}{10}\right) \tag{6}$$

$$0=\frac{EI}{l_1^3}\left(-6+2\theta_u l_1+8\Delta_4 l_1+2\theta_l l_1\right)-\frac{P}{l_1}\left(-\frac{1}{10}-\frac{\theta_u l_1}{30}+\frac{4\Delta_4 l_1}{15}-\frac{\theta_l l_1}{30}\right) \tag{7}$$

$$S=\frac{EI}{l_1^3}\left(12-6\theta_u l_1-12\Delta_3-6\Delta_4 l_1\right)-\frac{P}{l_1}\left(\frac{6}{5}-\frac{\theta_u l_1}{10}-\frac{6\Delta_3}{5}-\frac{\Delta_4 l_1}{10}\right) \tag{8}$$

$$-\frac{R_u\theta_u}{l_1}=\frac{EI}{l_1^3}\left(-6+4\theta_u l_1+6\Delta_3+2\Delta_4 l_1\right)-\frac{P}{l_1}\left(-\frac{1}{10}+\frac{2\theta_u l_1}{15}+\frac{\Delta_3}{10}-\frac{\Delta_4 l_1}{30}\right) \tag{9}$$

For the convenience of numerical calculation, the so-called end-fixity factor that characterizes the effect of beam-to-column end rotational restraints is introduced (Monforton and Wu, 1963):

$$r=\frac{1}{1+\frac{3EI}{Rl}} \tag{10}$$

Thus, the end rotational restraining stiffness R_l and R_u can be rewritten in terms of end-fixity factors r_l and r_u as

$$R_u=\frac{3EI}{l}\frac{r_u}{\left(1-r_u\right)} \tag{11a}$$

$$R_u=\frac{3EI}{l}\frac{r_u}{\left(1-r_u\right)} \tag{11a}$$

It can be found that there are five variables in Eqs. (5~9). Substituting Eqs. (11) into Eq. (5~9), the lateral stiffness of the column, S, can be expressed as

$$S=\beta(\phi,r_l,r_u)\frac{12EI}{l^3} \tag{12}$$

in which, ϕ is the applied load ratio and defined as

$$\phi = \sqrt{\frac{Pl^2}{EI}} = \pi\sqrt{\frac{P}{P_e}} \tag{13}$$

and $\beta(\phi, r_l, r_u)$ is a lateral stiffness modification factor of the column accounting for the effects of both axial load and column end rotational restraints, and can be expressed as

$$\beta = -\frac{1}{12}\frac{a_0 + a_1\phi^2 + a_2\phi^4 + a_3\phi^6 + a_4\phi^8 + +c\phi^{10}}{b_0 + b_1\phi^2 + b_2\phi^4 + b_3\phi^6 + c\phi^8} \tag{14}$$

where

$$a_0 = -132710400(r_i + r_j + r_i r_j) \tag{15a}$$

$$a_1 = 44236800 + 17694720\left(r_i + r_j\right) - 26542080 r_i r_j \tag{15b}$$

$$a_2 = 2474496\left(r_i + r_j\right) - 440064 r_i r_j - 5895240 \tag{15c}$$

$$a_3 = 158208 - 11218\left(r_i + r_j\right) + 732824 r_i r_j \tag{15d}$$

$$a_4 = 1127\left(r_i + r_j\right) - 947 r_i r_j - 1280 \tag{15e}$$

$$b_o = 44236800 - 1105920 r_i r_j \tag{15f}$$

$$b_1 = 2949120\left(r_i + r_j\right) - 1105920 r_i r_j - 5898240 \tag{15h}$$

$$b_2 = 158208 - 115968\left(r_i + r_j\right) + 80640 r_i r_j \tag{15c}$$

$$b_3 = 1136\left(r_i + r_j\right) - 992 r_i r_j - 1280 \tag{15e}$$

$$c = 3(1 - r_i)(1 - r_j) \tag{15f}$$

The stiffness modification factor β characterizes the relationship between column lateral stiffness and applied axial load. Similar to that discussed by Zhuang and Xu (2013), for the stiffness modification factor β defined in Eq. (14), an increase of the axial load would result in a decrease of the magnitude of β which consequently reduces the lateral stiffness of the column. A zero value of β denotes the column completely lost its lateral stiffness and will buckle in a lateral sway mode if no external support is provided. A negative value of β signifies that the column completely rely upon the external lateral support to sustain the applied load. With an adequate the external support, the column can sustain the applied load up to the level when magnitude of β

approaches the negative infinity which signifies the column reaches its rotational non-sway buckling strength. Two methods of evaluating β were proposed by Xu et al (2001). First, based on Euler-Bernoulli beam theory, a transcendental relationship (EB) between β and ϕ was derived. However, the transcendental relationship between β and ϕ resulted in the complexity of employing mathematical programming methods to obtain the maximum and minimum frame buckling strength when frames are subjected to variable loading. Subsequently, the Taylor series approximation of the transcendental relationship (TEB) was adopted for the reason of simplicity so that linear programing method can be employed to calculate the maximum and minimum frame buckling strength. Shown in Fig. 3 are the relationships between β and ϕ obtained from Eq. (14) and that of EB and TEB methods for different column end rotational restraints. Since the EB approach is theoretically derived without taking any approximation, it can be served as the benchmark in this comparison.

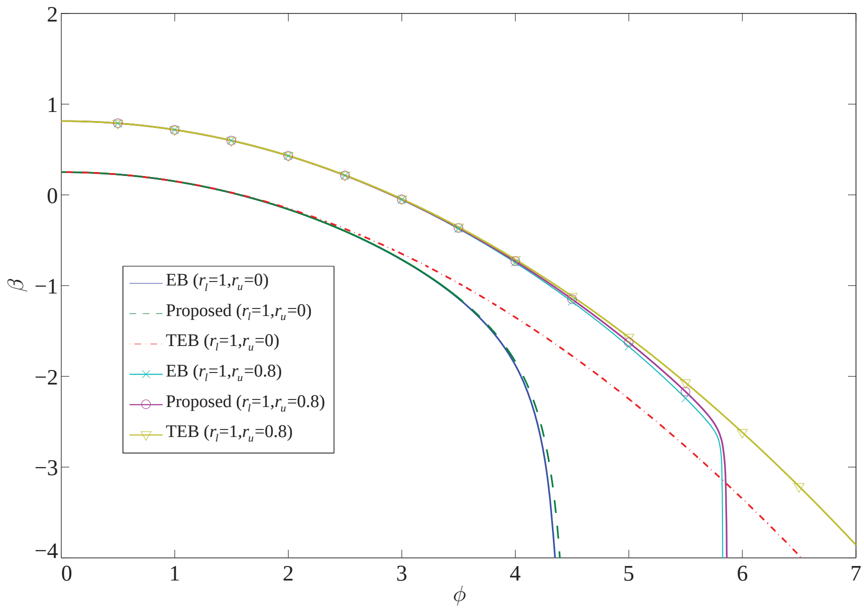

Fig. 3: Relationships between β and ϕ of different methods

It can be found from Fig. 3 that the sway lateral buckling strengths (β = 0) predicted by all three methods are close to each other. However, as the increase of the magnitude of ϕ, β values predicted by TEB method gradually deviate from that of EB method and the difference of β values between the two methods becomes significant. On the contrary, the result obtained from the proposed method is in good agreement with that of EB method even with substantial increase of the magnitude of ϕ. As the axially loaded columns in an unbraced frame can be buckled either in lateral sway or rotational non-sway mode depends on lateral stiffness of the frames, the predicted frame buckling strength would be inaccurate if the rotational non-sway buckling strength of a column cannot be accurately evaluated. Therefore, the proposed method need to predict the accurate results of critical value of ϕ for not only at β = 0 but also at

β = -∞ (negative infinity) which are associated with the sway and non-sway buckling mode, respectively. Once a critical value of ϕ calculated, the corresponding buckling strength can be obtained from Eq. (13).

Shown in Table 1 are the critical values of ϕ associated with rotational non-way buckling and lateral sway buckling for columns with different end restraints evaluated based on the three methods. It can be seen that results obtained from both TEB and the proposed methods are in good agreement with that of EB method while assessing the sway buckling strength of the columns. However, significant differences are observed when it comes to evaluate the buckling strength of braced columns. TEB method based on Taylor series approximation to evaluate β value is not applicable (NA) to calculate the rotational non-sway buckling strength of the columns. Consequently, TEB based method may result in erroneous predictions of the minimum and maximum unbraced frame buckling strength in the case when weak or heavily loaded columns tend to fail in rotational non-sway buckling mode when they are laterally braced adequately by strong or lightly loaded columns in the same storey. As buckling strength evaluated based on the proposed method are in good agreement with that of the EB for both the sway and non-sway columns. That being said, in evaluation of frame buckling strength subjected to variable loading, the proposed method would be able to overcome the deficiency associated with TEB method. In addition, the proposed method is simpler than EB method which would be a great advantage in terms of efficiency while evaluating the maximum and minimum frame buckling strength by means of mathematic programing algorithms.

Table 1: Critical value of ϕ for different column end restraints

Column end restraints	Method	Critical value of ϕ	
		Non-sway buckling (β = -∞)	Sway buckling(β = 0)
r_l=1, r_u=0	EB	4.49	1.57
	TEB	NA	1.58
	Proposed	4.55	1.57
r_l=1, r_u=0.2	EB	4.72	1.94
	TEB	NA	1.95
	Proposed	4.78	1.94
r_l=1, r_u=0.8	EB	5.83	2.90
	TEB	NA	2.92
	Proposed	5.87	2.91

2.2 The maximum and minimum buckling strength of unbraced frames

The concept of storey-based buckling states that an individual column cannot fail in lateral sway buckling mode alone without all of the other columns in the same storey also buckling in the same sway mode. Lateral sway buckling of unbraced frames is therefore a storey phenomenon involving the lateral stiffness interaction among columns in a storey. Based on the storey-based buckling concept, the criterion for multi-column buckling in a lateral sway mode is that the lateral stiffness of the storey vanishes. The problem of evaluating the minimum and maximum elastic buckling strength of a single storey frame subjected to variable loadings can be stated as a pair of minimization and maximization problems as follow (Xu et al. 2001):

$$\genfrac{}{}{0pt}{}{Minimum}{Maximum}: Z = \sum_{i=1}^{n} P_i \tag{16}$$

subjected to

$$\sum_{i=1}^{n} S_i = 12E \sum_{i=1}^{n} \beta_i \frac{I_i}{L_i^3} = 0 \tag{17a}$$

$$P_{il} \le P_i \le P_{iu} = \frac{\pi^2 E I_i}{K_i^2 L_i^2} \left(i = \ ,2,\ldots,n\right) \tag{17a}$$

where the applied column load P_i is the variable to be solved in the problem. The objective function Eq. (16) corresponds to either the minimum or the maximum elastic frame buckling strength, as given by the sum of individual column loads. The storey-based lateral instability condition imposed on the frame is defined by Eq. (17a), in which β_i is the lateral stiffness modification factor defined in Eq. (14) and is a function of the applied load. The lower bound imposed on each applied column load can be specified by the invariant portion of the applied load, while the upper bound, $\pi^2 E I_i / K_i^2 L_i^2$, is imposed to ensure that the magnitude of the applied load will not exceed the critical load associated with rotational non-sway buckling of the individual column. The factor K_i is the effective length factor of the column associated with rotational non-sway buckling and is evaluated as the following (Newmark 1949, Xu 2003):

$$K_i^2 = \frac{\left[\pi^2 + \left(6-\pi^2\right) r_{iu}\right] \times \left[\pi^2 + \left(6-\pi^2\right) r_{il}\right]}{\left[\pi^2 + \left(12-\pi^2\right) r_{iu}\right] \times \left[\pi^2 + \left(12-\pi^2\right) r_{il}\right]} \tag{18}$$

Unlike the minimization and maximization problems discussed by Xu et al (2001), which can be solved by linear programming method because there is a linear relationship between the applied load P_i and the stiffness modification factor β_i. The relationship defined in Eq. (17a), however, is nonlinear and contains polynomials only. Thus, the minimization and maximization problems stated in Eqs. (16) and (17) will be able to be solved by efficient gradient-based nonlinear programming algorithm. In this study, sequential quadratic programming algorithm is employed to solve the problems.

3. NUMERICAL EXAMPLES

The stability of four single-storey unbraced steel frames subjected to variable loading as shown in Fig. 4 were investigated by Xu (2003) based on expressions similar to that of Eqs. (16) and (17). However, because the column stiffness was evaluated by EB method which resulted in the constraint shown in Eq. (17a) containing transcendental functions, it was too complicated to directly use gradient-based nonlinear programming algorithms to solve the problems. Consequently, the problems of Eqs. (16) and (17) were reformulated by combining the objective function Eq. (16) and the stability constraint Eq. (17a) in which an artificial penalty coefficient was applied to Eq. (17a). The reformulated problems were then solved by a less efficient non-gradient-based algorithm.

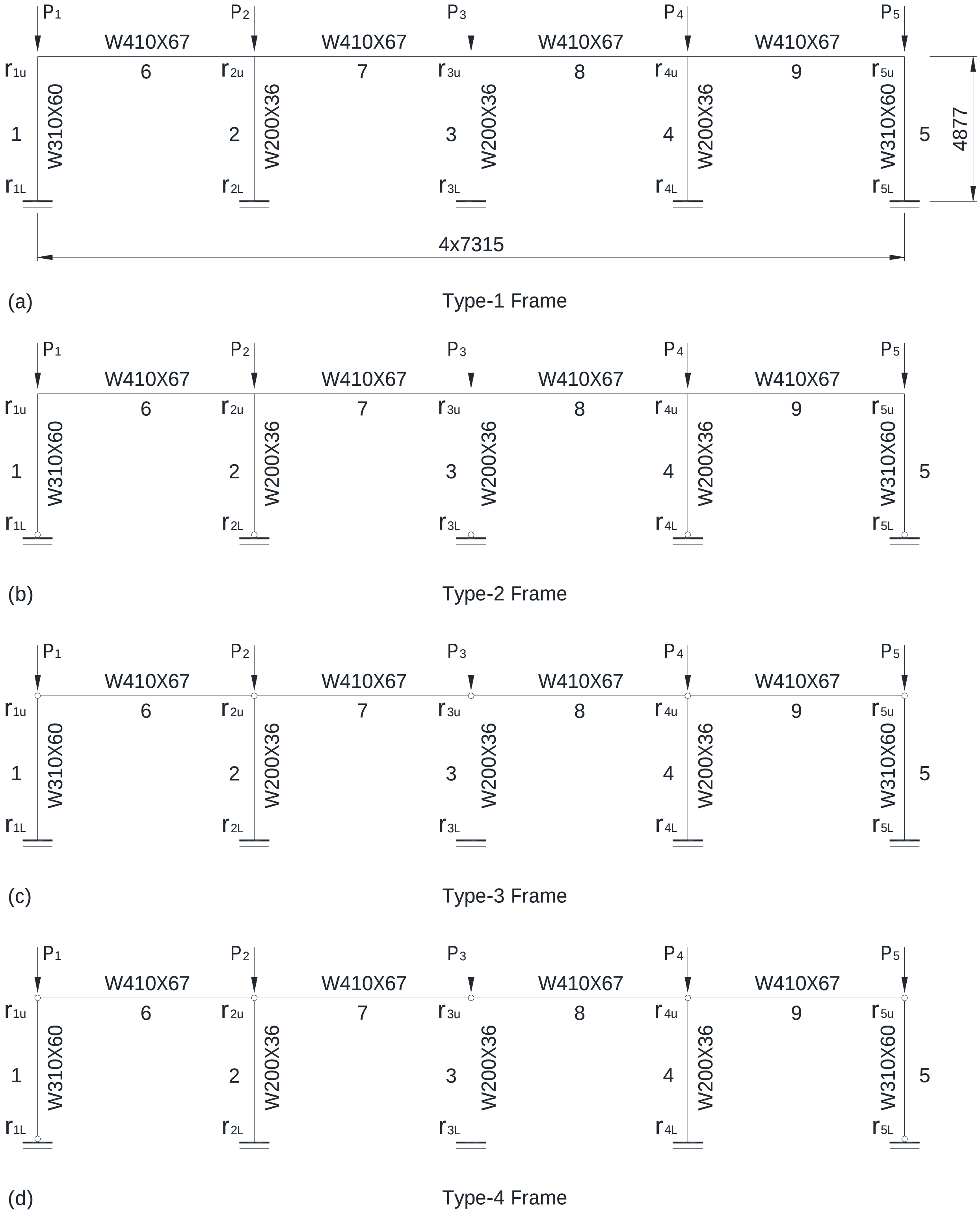

Fig. 4: One-storey four-bay unbraced steel frames

In this study, the problems defined in Eqs. (16) and (17) for the four frames are solved by the sequential quadratic programming method For the steel frames shown in Fig. 4, Young's modulus of steel E is taken to be 2×105MPa. The moments of inertia for the exterior columns are I_1=I_5=129×106mm^4, while I_2=I_3=I_4=34.1×106mm^4 for the interior columns. The moments of inertia of the beams are I_6=I_7=I_8=I_9=245×106mm^4. All the connections in the four frames are either rigid or pinned connections. The lower and upper bounds for the applied column loads are P_{il} = 0 and P_{iu} = $\pi^2 EI_i / K_i^2 L_i^2$, respectively, where the column effective length factors K_i are defined in Eq. (12). The end-fixity factors for each column of the four frames are listed in Table 2 (Xu 2003).

The minimum and maximum buckling strength of the four steel frames obtained from the sequential quadratic programming in which the lateral stiffness of columns are evaluated by the proposed method are presented in Table 3. For the reason of comparison, the results obtained from a non-gradient-based algorithm (NLP-NG) by Xu (2003) and linear programming (LP) by Zhuang and Xu (2013) in which the stiffness are calculated based on EB and TEB method, respectively, are also listed in Table 3. The upper bounds of the column applied loads of the frames evaluated based on P_{iu} = $\pi^2 EI_i / K_i^2 L_i^2$ are also listed in the table. The relative differences among methods are presented in Table 4.

Table 2: Column end-fixity factors

End-fixity Factor	Frame			
	Type-1	Type-2	Type-3	Type-4
r_{1l}	1.0	0.0	1.0	0.0
r_{1u}	0.717	0.717	0.0	0.0
r_{2l}	1.0	0.0	1.0	1.0
r_{2u}	0.95	0.95	0.0	0.0
r_{3l}	1.0	0.0	1.0	1.0
r_{3u}	0.95	0.95	0.0	0.0
r_{4l}	1.0	0.0	1.0	1.0
r_{4u}	0.95	0.95	0.0	0.0
r_{5l}	1.0	0.0	1.0	0.0
r_{5u}	0.717	0.717	0.0	0.0

Table 3: Frame buckling loads and strengths

Frame Type	Loading Pattern	Method	P_1 (kN)	P_2 (kN)	P_3 (kN)	P_4 (kN)	P_5 (kN)	ΣP_i (kN)
1	Upper bound	P_{iu}	34394	10869	10869	10869	34394	
	minimum	LP	0	8213.5	8213.5	8213.5	0	24641
		NLP-NG	717	7420	7420	7420	717	23694
		NLP-G	0	2144	10869	10869	0	23882
	maximum	LP	12858	0	0	0	12858	25716
		NLP-NG	12664	0	0	0	12664	25328
		NLP-G	12735	0	0	0	12735	25470
2	upper bound	P_{iu}	17197	5434	5434	5434	17197	
	minimum	LP	0	2010	2010	2010	0	6030
		NLP-NG	0	0	0	4485	0	4485
		NLP-G	0	0	4523	0	0	4523
	maximum	LP	3227	0	0	0	3227	6454
		NLP-NG	3178	0	0	0	3178	6356
		NLP-G	3179	0	0	0	3179	6454
3	upper bound	P_{iu}	21411	5660	5660	5660	21411	
	minimum	LP	1515	1515	1515	1515	1515	7575
		NLP-NG	0	1244	0	4655	0	5899
		NLP-G	0	0	0	5043	0	5043
	maximum	LP	1515	1515	1515	1515	1515	7575
		NLP-NG	2695	695	695	695	2695	7475
		NLP-G	2679	708	708	708	2679	7482
4	upper bound	P_{iu}	10706	5660	5660	5660	10706	
	minimum	LP	0	717	717	717	0	2151
		NLP-NG	0	0	0	2047	0	2047
		NLP-G	0	0	0	2050	0	2050
	maximum	LP	1290	0	0	0	1290	2580
		NLP-NG	0	0	0	0	2580	2580
		NLP-G	1290	0	0	0	1290	2580

Table 4: Difference of minimum buckling strengths between different methods

Pattern	Difference	Type-1	Type-2	Type-3	Type-4
Maximum	LP to NLP-NG	+1.5%	+1.5%	+1.3%	0%
	NLP-G to NLP-NG	+0.56%	+1.5%	0.09%	+0.15%
Minimum	LP to NLP-NG	+4%	+34%	+28%	+5%
	NLP-G to NLP-NG	+0.79%	+0.85%	-0.15%	0%

Comparing the results shown in Table 3 and 4, it can be found that the minimum and maximum frame buckling strengths obtained from the nonlinear programming problems defined in Eqs. (16) and (17) based on the proposed gradient-based method are very close to the that solved by the non-gradient-based algorithm. The results also suggest the proposed method is reliable in evaluating the minimum and maximum buckling strength of unbraced steel frames subjected to variable loadings as the difference between the results of NLP-G and NLP-NG are less than 2%. It is also noted that the frame buckling strength evaluated based on the proposed method (NLP-G) are slightly greater than that of NLP-NG method except one case, which indicates that column stiffness modification factors βi are slightly overestimated by the proposed method compared to that of EB method. The one case that frame buckling strength yield by the proposed method is less than that of NLP-NG method is the minimum frame buckling strength of Type-3 frame, in which the strength obtained from the proposed method (NLP-G: ΣP_i = 5043 kN) is less than that obtained from the nonlinear programming problem solved by non-gradient-based algorithm (NLP-NG:ΣP_i = 5899 kN). In this case, it appears that the non-gradient-based algorithm did not find a global minimum other than a local one.

It is also observed that the load patterns associated with the frame buckling strengths obtained from the proposed and NLP-NG methods may not be the same, which indicates there may be more than one load patterns associated with the minimum or maximum frame buckling strength in the case of variable loading. As discussed by Xu (2003), the load patterns associated with the minimum and maximum frame buckling strength depend on the presence of strong and weak columns, as defined by large and small lateral stiffness capacity, respectively. The load patterns associated with the minimum frame buckling strength tends to maximize loads on the weak columns and minimize the loads on the strong columns and vice versa for that associated with the maximum frame buckling loads. The results shown in Table 3 are in agreement in with that observation.

Meanwhile, it is observed from Table 4, the relative differences between the results of LP and NLP-NG methods are much greater than that between the results of NLP-N and proposed (NLP-G) methods. That being said, the results obtained from the proposed method are more accurate than that obtained from the linear programming method. Also, it can be found that for all these four type frames, LP method overestimates the minimum frame buckling strength considerably. In addition, LP method failed to detect the minimum frame buckling strength of Type-3 frame. In practice, the minimum frame buckling strength is far more important to engineering practitioners than the maximum one since the minimum frame buckling strength associated with the worst load pattern under which the framed structure will fail with the minimum amount of applied load. Therefore, despite the simplicity, LP method in which column lateral stiffness are evaluated based on the Taylor series approximation of Euler-Bernoulli theory (TEB) may not be suitable for engineering practice as it may result in unsafe designs. From this perspective, the proposed method is more preferable and recommended for the investigation of frame stability subjected to variable loading.

4. CONCLUSIONS

In this study, an alternative method was proposed to evaluate the lateral stiffness of an axially loaded column by modeling a column with two cubic Hermite elements. The accuracy of the proposed method on column rotational non-sway and lateral sway buckling strength are investigated for columns with different end rotational restraints and bracing conditions. The results shown that the proposed method are in good agreement with that obtained from the method based on Euler-Bernoulli beam theory proposed by Xu et al (2001). Unlike the method based on Euler-Bernoulli beam theory in which the expression of lateral stiffness of axially loaded column involves transcendental functions of applied load, the proposed equation only contains polynomials, which is certainly simpler than the transcendental functions especially when the nonlinear programming algorithms are employed to find the maximum and minimum buckling strength of unbraced frames subjected to variable loading.

Four single storey unbraced steel frames subjected to variable loadings are investigated in this study. The investigation demonstrated that the maximum and minimum frame buckling strength obtained based on the proposed method with use a gradient-based algorithm are can yield accurate results that are in good agreement with that evaluated based on Euler-Bernoulli theory with use of a non-gradient-based algorithm (Xu, 2003). It is also found in the examples that the lateral stiffness of axially loaded column may be overestimated when the Taylor series approximation of Euler-Bernoulli beam theory was employed. Consequently, the frame buckling strength evaluated based on the linear programming formulation may be overestimated which can result in unsafe design. In that respect, stability capacity of unbraced frame subjected to variable loading should be evaluated in accordance with the nonlinear programming based formulation and solved by the proposed method.

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