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Buckling analysis of structures subject to combined loading including acceleration forces[a]

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ABSTRACT

The structures of concern in this study are subject to two types of forces: dead loads from the acceleration imposed on the structures as well as the installed operation machines and the additional adjustable forces. We wish to determine the critical values of the adjustable forces when buckling of the structures occurs. The mathematical statement of such a problem gives rise to a constrained eigenvalue problem (CEVP) in which the dominant eigenvalue is subject to an equality constraint. A numerical algorithm for solving the CEVP is proposed in which a trial-and-error method is employed to identify an interval embracing the target eigenvalue. The algorithm is applied to three engineering application examples including the critical load of a cantilever beam subject to its own body force, and buckling loads of two plane structures when body force is present. The accuracy is demonstrated using the first example whose classical solution exists. The significance of the equality constraint in the CEVP is shown by comparing the solutions without the constraint on the eigenvalue. Effectiveness and accuracy of the numerical algorithm are presented.

1. INTRODUCTION

Buckling has been one of the main concerns in structure design against catastrophic failure for a long time. Naturally the topic has attracted a large group of researchers and engineers in the past rendering a rich source of articles in the area. A few textbooks in theoretical settings as well as numerical practices have been published and used in

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academia, such as Timoshenko and Gere (2009), Bathe (1996) and Cook *et al*. (2007), which provide a good source of references in the related fields. Further, the numerical procedures of finding the buckling loads have been implemented in a few commercial codes for engineering practices, for example, ANSYS (ANSYS Inc., 2012), ADINA (ADINA R & D Inc., 2012), MARC (MSC Software 2013), and ABAQUS (Simulia 2011). The study regarding the buckling of the elastic object subject to gravity as well as other applied loads has received much less attention. Roberts and Azizian (1984) and Roberts and Burt (1985) investigate the lateral buckling of an elastic I-beam subject to uniformly distributed load using energy method. Influence of such parameters as sectional warping rigidity, location of applied load with respect to the shear center is thoroughly studied. Dougherty (1990, 1991) considers the lateral buckling of an elastic beam subject to uniformly distributed load as well as a central point load and end moments. In the studies, gravity load of the beam is modeled as a uniformly distributed load applied on the top surface of the beam. A numerical approach is employed to solve for the critical load for the beam.

The loads applied on the beam in the studies by Kerstens (2005) and Cheng *et al*. (2005) appear to be proportional in that the point force and the uniformly distributed load, for example, vary at the same rate, if necessary. In this currently study, gravitational load and other applied forces are non-proportional. Thus the buckling problem under the influence of gravity is formulated as a constrained eigenvalue problem. Kerstens (2005) provides a review of methods employed in solving constrained eigenvalue problems. Cheng *et al*. (2005) present a classic study of the buckling of a thin circular plate. In the study, Ritz method is employed to solve the first buckling load of the circular plate with boundary fixed. The only load is the in-plane gravity. Kumar and Healey (2010) present a study of stability of elastic rods. The generalized eigenvalue problem consists of a set of constraint equations imposed on the nodal displacements of the model. There is no constraint on the eigenvalue itself. Efficient numerical methods are presented to solve the first few lowest natural eigenvalues. Zhou (1995) examines an algorithm for the design optimization of structure systems subject to both displacement as well as eigenvalue (natural frequency) constraints. An iterative algorithm based on Rayleigh Quotient approximation is shown to be efficient in solving the dual constraint eigenvalue problems.

In this paper, the problem to be tackled is given and formulated in mathematical form in Section 2. The deviation of the current problem from the others is disclosed. It is shown that addressing the current problem using the usual treatment would lead significant errors. Section 3 presents a simple algorithm for solving the problem efficiently. The proposed algorithm is tested using three numerical examples in Section 4. It is seen from the examples that the proposed algorithm has achieved excellent accuracy.

2. MATHEMATICAL STATEMENT OF THE CURRENT PROBLEM

Conventionally the buckling load of a structure can be determined by solving the following eigenvalue problem.

$$[\mathbf{K} + \lambda \mathbf{K}_f]\mathbf{U} = \mathbf{0}. \quad (1)$$

where **K** is the usual stiffness matrix of the structure, λ the eigenvalue or load factor (LF), **U** the nodal displacement vector, and $\mathbf{K}_f$ the stiffness matrix of the same structure due to stress stiffening from an externally applied force *f* set at an arbitrary reference magnitude given as follows.

$$\mathbf{K}_f = \int \mathbf{G}^T \mathbf{S} \mathbf{G} \, dV \quad (2)$$

where **G** and **S** are, respectively, modified strain-displacement and stress matrices (Bathe 1996, Cook *et al*. 2007). Note that in case of line elements, the stress matrix contains only a component **S** = [σ_x] where σ_x is the axial stress in the elements. Thus, Eq. (2) becomes

$$\mathbf{K}_f = \int \mathbf{G}^T \sigma_x \mathbf{G} \, dV \quad (3)$$

Likewise, the higher-order Green-strain-displacement matrix for line elements is given as below.

$$\mathbf{G} = \frac{d}{dx}\mathbf{N} \quad (4)$$

where the shape function matrix **N** contains the usual linear and cubic Hermitian interpolation functions for bar and beam elements, respectively (Cook *et al*. 2007). It is understood that for a non-trivial solution to exist, the determinant of the multiplier matrix in Eq. (1) must be zero.

$$\|\mathbf{K} + \lambda \mathbf{K}_f\| = \mathbf{0}. \quad (5)$$

Once the eigenvalues are found, the critical buckling load f_c of the structure is given as follows.

$$f_c = \lambda_1 f, \quad (6)$$

where λ_1 is the lowest eigenvalue known as critical load factor.

As depicted in Figure 1, a deformable object is loaded with a reference force *f* while being subject to a given acceleration motion a_0. As a result, there are two stress stiffening matrices due to the applied load and the acceleration force, $\mathbf{K}_f$ and $\mathbf{K}_{a0}$, respectively. It is our goal to determine the buckling load of the structure while it is under the given acceleration. Thus for the current problem an eigenvalue system to be solved may be given below.

$$[\mathbf{K} + \lambda(\mathbf{K}_f + \mathbf{K}_{a0})]\mathbf{U} = \mathbf{0}, \tag{7}$$

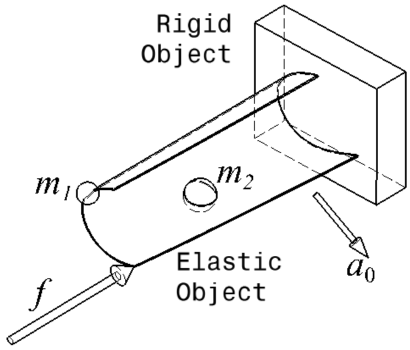

Figure 1 Schematic of the present problem – an object attached to a rigid support and subject to both external force and acceleration a_0.

After the eigenvalue problem is solved, the critical buckling load of the structure can be determined using Eq. (6). Meanwhile, there would be a "critical acceleration" which in combination with the critical load would put the structure in an unstable state. The acceleration under the critical condition a_c is determined as follows.

$$a_c = \lambda_1 a_0. \tag{8}$$

Unless $\lambda_1 = 1$, we have $a_c \neq a_0$. Clearly, the above methodology does not provide the correct solution to the problem.

Consequently a proper method is required to solve the eigenvalue problem so that the acceleration remains at the fixed value a_0 when buckling occurs. Consider the following constrained eigenvalue problem (CEVP).

$$[\mathbf{K} + \lambda(\mathbf{K}_f + \alpha\mathbf{K}_a)]\mathbf{U} = \mathbf{0}, \tag{9}$$

subject to

$$\alpha a \lambda_1 = a_0, \tag{10}$$

where **K**, $\mathbf{K}_f$ are the same matrices as before, $\mathbf{K}_a$ the stress stiffening matrix using a reference acceleration *a*, and *α* an *unknown* participation factor. Of concern is the buckling load f_c of the structure while the acceleration remains at a_0. Since *a* is a reference number, we may choose *a* = 1 for convenience.

Note that other constrained eigenvalue problem exists (Kerstens 2005, Kumar and Healey 2010) in which equality constraints are imposed on the eigenvectors so some nodal displacements in the model are deformed in a specific way.

3. NUMERICAL ALGORITHM

For a given structure, the total stiffness matrix **K** can be readily formed first. The stress stiffening matrix $\mathbf{K}_f$ can be obtained by using the stress stemming from an arbitrarily chosen reference force *f* corresponding to the applied load which remains the same throughout the following numerical scheme. To obtain the stress stiffening matrix $\mathbf{K}_a$ due to acceleration, we may choose $a = 1$ for convenience. In the following numerical scheme, a series of values for the participation factor α_i is used in solving the following eigenvalue problem.

$$[\mathbf{K} + \lambda(\mathbf{K}_f + \alpha_i \mathbf{K}_a)]\mathbf{U} = \mathbf{0}, \qquad (11)$$

Therefore, a series of acceleration a_i is obtained via the following relation.

$$\alpha_i \lambda_1 = a_i, \qquad (12)$$

In the scheme seen in Figure 2, the eigenproblem is solved until the target value a_0 falls within the interval: $a_i \le a_0 \le a_{i+1}$.

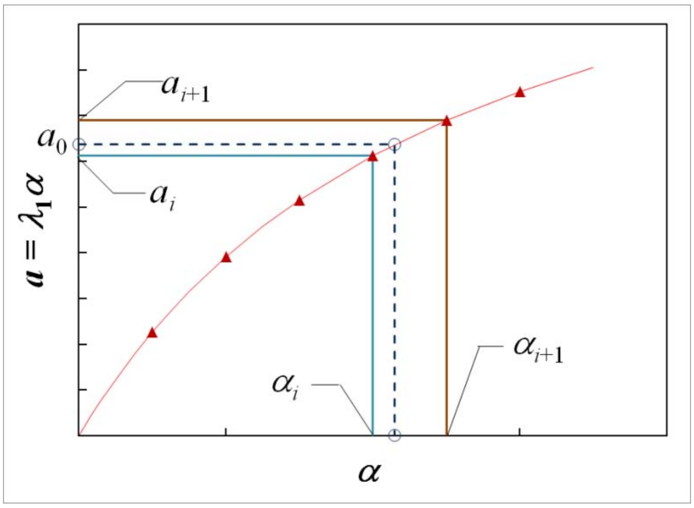

Figure 2 The trapping scheme for finding unknown α.

Let us introduce a natural coordinate , –1 ≤ ≤ +1. From the following linear interpolation, we can determine the natural coordinate corresponding to the target value a_0.

$$a_0 = \frac{1}{2}(1-\xi)a_i + \frac{1}{2}(1+\xi)a_{i+1}. \qquad (13)$$

Or,

$$\xi = (2a_0 - a_i - a_{i+1})/(-a_i + a_{i+1}). \qquad (14)$$

Upon substituting this natural coordinate into the following interpolation equation, the unknown participation factor can be determined.

$$\alpha = \frac{1}{2}(1-\xi)\alpha_i + \frac{1}{2}(1+\xi)\alpha_{i+1}. \tag{15}$$

It is worth mentioning that linear interpolation is used in the above calculation with a proper selection of the increment used in α_i. The result obtained certainly can be improved if quadratic interpolation is used.

The eigenproblem Eq. (9) is solved one more time using the participation factor found from Eq. (15). The eigenvalue found together with the participation factor in Eq. (15) constitute the solution to the constrained eigenvalue problem. The proposed algorithm can be easily implemented in ANSYS APDL (ANSYS Inc., 2012). In the following section, we use three examples to demonstrate the accuracy and efficiency of the algorithm presented here.

4. APPLICATION EXAMPLES

All the examples presented in this section are two-dimensional; the algorithm can be extended to three-dimensional cases easily. A theoretical solution in approximate form exists for the first example, which serves as the guide for validating the accuracy of the proposed algorithm. In the other two examples, the purpose is to demonstrate the efficiency of the numerical algorithm. It is not intended to identify the worst case scenario.

4.1 Buckling of a Beam Subject to Constant Acceleration

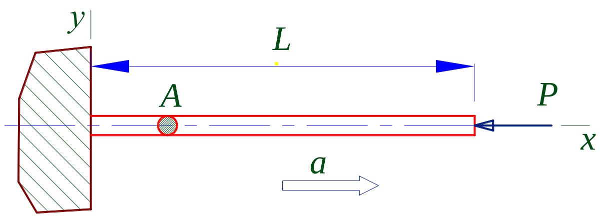

Figure 3 An elastic beam subject to a constant acceleration and an axial force *P*.

As depicted in Figure 3, an elastic beam is subjected to a point force *P* at the free end as well as constant acceleration *a*. The theoretical solution of the buckling load when the acceleration is the gravity g is given approximately as follows (Timoshenko and Gere 2009).

$$P_{cr} \approx \frac{\pi^2 EI}{4L^2} - 0.3\rho ALg \tag{16}$$

where *EI* is the beam's flexural rigidity and *L* the length of the elastic beam. Note also that the beam would buckle due to its own weight if the following equation holds

(Timoshenko and Gere 2009).

$$(\rho ALg)_{cr} \approx \frac{7.837EI}{L^2} \quad (17)$$

Assuming steel is used (E = 200 GPa, ρ = 7,890 kg/m^3), acceleration is $a = g$ = 9.81 m/s^2 and the geometric properties of the beam are: L = 5 m, A = 1.58×10^{-4} m^2 and I_{zz} = 2.725×10^{-9} m^4, Eq. (16) yields the following exact solution.

$$P_{cr} = 35.45 \text{ N} \quad (18)$$

To use the proposed numerical scheme for a twenty-five two-dimensional beam elements for the beam in Figure 3, a MATLAB code is developed. Note that some of codes in the text by (Kwon and Bang, 2000) come handy for this endeavor. In the calculation, the reference force and acceleration chosen are: P = 10 N and a = 1 m/s^2.

Table 1 reveals a few calculation steps used to contain the target acceleration g = 9.81 m/s^2 between the 4th and the 5th steps. Therefore, upon using $a_4 = 9.7455$; $a_5 = 10.3071$ in Eq. (14) the natural coordinate corresponding to the target acceleration g is $\xi = -0.7701$. The participation factor determined through Eq. (15) is $\alpha = 2.7787$.

Table 1 Numerical calculation for finding the participation factor for the elastic beam in Figure 3.

No.	P, N	α	λ	a, m/s^2
1	10	2	3.9121	7.8242
2		2.25	3.7813	8.5079
3		2.5	3.6588	9.1470
4		**2.75**	3.5438	**9.7455**
5		**3**	3.4357	**10.3071**

With the combination of P = 10 N and α = 2.7787, the eigenvalue problem Eq. (9) is solved once more which gives λ_1 = 3.5311. Consequently, the beam is subject to the acceleration $a = \alpha\lambda_1$ = 9.812 m/s^2, which is the gravity. And, the buckling load for the beam is: $P_{cr} = P\lambda_1 = 35.31$ N which is within 0.2% of the exact solution.

If we used Eq. (7) with the reference force and acceleration, P = 10 N and a = 1 m/s^2, to determine the buckling load of the beam, we would have obtained λ_1 = 4.5338. Thus, the critical force at buckling would have been $P_{cr} = \lambda_1 \times 10 \text{ N} = 45.34 \text{ N}$ while the beam is subject to an acceleration $a = \lambda_1 \times 1$ m/s^2 = 4.5338 m/s^2.

4.2 Buckling of a Truss Structure

Figure 4 shows a plane truss of a simplified crane. It has three point masses at three different locations. The mass m_1 = 4,000 kg at point F represents the mass of a counterweight, while $m_2 = m_3$ = 2,000 kg are the masses of a control unit of the crane at points D and E. The truss is constrained so no translational movement at points A and B.

Note that all members of the truss and the two cables are made of steel (E = 200 GPa, ρ = 7,890 kg/m^3) with circular cross-section r_1 = 12.5 mm. It is our intent to determine the maximum load W_c that the truss can carry at point F prior to buckling.

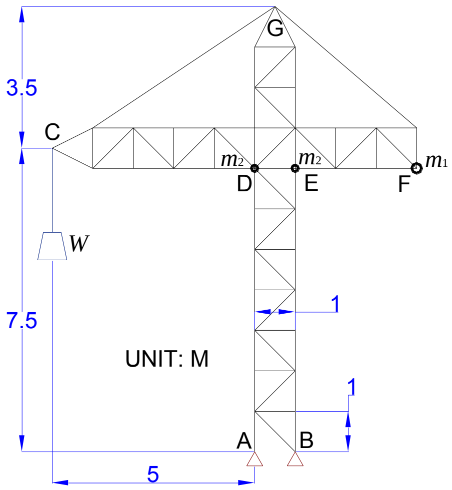

Figure 4 A plane truss with three concentrated masses at points D, E and F carries a payload W at C.

In this example, 74 link elements are used for the bars and cables. Each node has two translational degrees of freedom. Three point elements are used to model the masses at points D, E and F. Table 2 shows the numerical calculation using a code written in MATLAB. Therefore, the target acceleration is enclosed in the interval between a_4 = 9.3918 and a_5 = 10.6022.

Table 2 Numerical calculation of finding the buckling load for the plane truss in Figure 4.

No.	W, KN	α	λ	a, m/s^2
1	10.0	0.03	173.91	5.2173
2		0.04	167.72	6.7088
3		0.05	161.93	8.0965
4		**0.06**	156.53	**9.3918**
5		**0.07**	151.46	**10.6022**

From Eqs. (14) and (15) the natural coordinate corresponding to the gravitational acceleration g and the participation factor are, respectively:

$$\xi = 0.3377 \text{ and } \alpha = 0.06338.$$

The eigenvalue problem Eq. (9) is solved one final time using W = 10.0 KN and α = 0.06338 which results in λ_1 = 154.78. The downward acceleration the plane truss subject to is $a = \alpha\lambda_1$ = 9.810 m/s^2. And, the buckling load for the beam is: $W_c = W\lambda_1 = 1547.8$ KN. Note that upon solving the eigenvalue problem from Eq. (1) using the stress stiffening matrix $\mathbf{K}_f$ of the truss structure stemming from the reference force W = 10.0 KN *and* the gravitational acceleration g = 9.81 m/s^2, the eigenvalue is λ_1 = 4.553. This indicates that the buckling load would have to be $W_c = W\lambda_1 = 45.53\ \mathrm{KN}$, which is only 3% of the buckling load using the current algorithm. To cause the truss structure to buckle at this load the gravitational acceleration would have to be $a = g\lambda_1$ = 44.66 m/s^2.

4.3 Buckling of a Plane Frame on an Accelerating Vehicle

In the third example, a plane steel (E = 200 GPa, ρ = 7,870 kg/m^3) frame installed on a vehicle is subject to a known point force F_D = 1,500 N at point D. In addition there are two known masses m_1 = 200 kg, m_2 = 300 kg at points C and E. The horizontal beams of the frame are of a wide-flange cross-section, while the vertical beam is of C-channel. The cross-sections of beams are: A-A: $A_1 = 484\times10^{-6}$ m^2 and $I_1 = 4.205\times10^{-8}$ m^4, and B-B: $A_2 = 384\times10^{-6}$ m^2, and $I_2 = 9.260\times10^{-8}$ m^4. It is of interest to know the critical horizontal acceleration a_c of the vehicle when the 2D frame buckles. Note the gravity is g = 9.81 m/s^2 and the frame is hinge-supported at points A and G as shown in Figure 5.

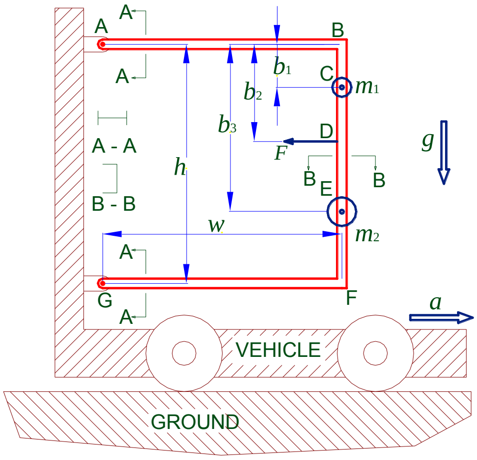

Figure 5 A traveling vehicle with a plane frame carrying two masses m_1 and mass m_2 and subject to a force F_D.

Let us introduce a scaling factor β so that

$$\beta = \frac{F_D}{g} \tag{19}$$

which simply correlates the magnitudes of the two given quantities. Thus, β = 152.91 for the present case. To tackle this example, Eqn. (9) is modified to account for the various loads as follows.

$$\left[\mathbf{K} - \lambda[\alpha g_1(\mathbf{K}_{g1} + \beta\mathbf{K}_{F1}) + a\mathbf{K}_a]\right]\mathbf{U} = \mathbf{0}, \tag{20}$$

where both g_1 and a are arbitrarily chosen magnitudes corresponding to gravity g and the horizontal acceleration and $F_1 = \beta g_1$. Furthermore, $\mathbf{K}_{g1}$, $\mathbf{K}_{F1}$ and $\mathbf{K}_a$ are, respectively, the individual stress stiffening matrices due to g_1, F_1 and a alone. The constraint equation (10) becomes

$$\alpha g_1 \lambda_1 = g. \quad (21)$$

Table 3 Numerical calculation for finding the participation factor for the plane frame in Figure 5.

No.	g, m/s^2	α	λ	a, m/s^2
1	1.0	1.5	6.4605	9.69075
2		1.6	6.094	9.7504
3		**1.7**	5.7667	**9.80339**
4		**1.8**	5.4727	**9.85086**
5		1.9	5.2072	9.89368

For the numerical study, the dimensions used for the frame are h = 6 m, w = 4 m, b_1 = 1.5 m, b_2 = 2 m and b_3 = 4.5 m. Using g_1 = 1 m/s^2 and a = 1 m/s^2 in the proposed algorithm, the critical participation factor found from Eqs. (14) and (15) is α_{cr} = 1.713 which is between steps 3 and 4 in Table 3.

With this α_{cr} the eigenvalue problem Eq. (20) gives λ_1 = 5.726. Therefore,

$$\alpha_{cr} g_1 \lambda_1 = 9.81 \text{ m/s}^2,\ F_D = \alpha_{cr} g_1 \lambda_1 \beta = 1499.77 \text{ N, and } a_c = \lambda_1 a = 5.726 \text{ m/s}^2. \quad (22)$$

It is seen that constraint Eq. (21) is satisfied. Buckling of the plane frame occurs when the horizontal acceleration is $a_c = 5.726 \text{ m/s}^2$ at which the applied force is exactly as specified, F_D = 1,500 N. The deformed shape of the frame in the first buckling mode is shown in Figure 6. Note that the setting for this model is not symmetric.

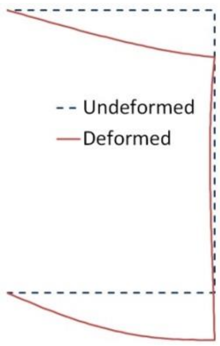

Figure 6 The first buckle mode of the plane frame in Figure 5.

5. CONCLUSION

The determination of the buckling load of an elastic structure in the presence of gravitational force is formulated as an eigenvalue problem subject to an equality constraint correlating an unknown participation factor and the given acceleration. A methodology of solving the constrained eigenvalue problem is presented. In the numerical algorithm, the eigenvalue problem is solved incrementally until the desired participation factor falls within an interval. Interpolation is employed to extract the accurate solution for the unknown. Three examples are used to demonstrate the accuracy of the numerical algorithm. Among them, one has an approximate theoretical solution. The solution predicted by the proposed algorithm is in excellent agreement with the theoretical solution. From the other two examples involving two-dimensional truss and frame, it is shown that the critical buckling loads predicted from the proposed algorithm are lower than those from the usual procedure by a relatively significant amount. The procedure involves some manual intervention and is laborious. It is necessary to develop an automatic numerical scheme for the problem in the future.

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