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Generalized Finite Element Analysis Using the Preconditioned Conjugate Gradient Method

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ABSTRACT

In this paper, we propose a generalized finite element analysis technique using the preconditioned conjugate gradient method. The proposed methodology is able to generate enrichment functions for problems where limited a-priori knowledge on the solution is available and to utilize a preconditioner and initial guess of good quality with a small addition of computational cost. Thus, it is very effective to analyze problems where a complex behavior is locally exhibited. Several numerical experiments are performed to confirm its effectiveness and show that it is computationally more efficient than the analysis using direct solvers such as the Cholesky factorization.

1. INTRODUCTION

The conjugate gradient method (CGM) is a semi-iterative type method which is efficient for solving a linear system of a large and sparse matrix. This method has not been used extensively in the finite element analysis since the condition number of the stiffness matrix significantly affects convergence rate and it may even become less computationally efficient than the direct solver such as the Cholesky or LU factorization method. To overcome this shortcoming, many different types of preconditioner has been developed to improve the conditioning of the linear system. (Heath 2002) However, there exists no universal preconditioner, and its creation is computationally expensive in general.

Recently, the authors proposed a generalized finite element method with global-local enrichment functions. ($GFEM^{gl}$) This method utilizes customized enrichment functions for applications where limited a priori knowledge about the solution is available. (Duarte and Kim 2008) The procedure involves the solution of local boundary value problems using boundary conditions from a coarse global problem. The local solutions are in turn used to enrich the global solution space using the partition of unity methodology. In this approach, a preconditioner and initial guess of good quality can be constructed with a small addition of computational cost, thus the use of the

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preconditioned CGM (PCGM) with this approach can result in significant saving in computational cost. Therefore, in this paper, the application of the PCGM to the proposed GFEM is explored, and its effectiveness is investigated in comparison with direct methods.

2. GENERALIZED FINITE ELEMENT METHOD WITH GLOBAL-LOCAL ENRICHMENTS

The core concept of the *GFEM*^{gl} is to use numerically generated enrichment functions that are customized for applications with limited a priori knowledge about the solution. (Kim et al. 2010) The procedure involves the solution of local boundary value problems using boundary conditions from a coarse global problem. The local solutions are in turn used to enrich the global solution space using the partition of unity methodology. The global shape function $\boldsymbol{\varphi}_{\alpha}(\mathbf{x})$ enriched with the local solution $\mathbf{u}_{L}^{\alpha}(\mathbf{x})$ is given by Eq. (1).

$$\boldsymbol{\varphi}_{\alpha}(\mathbf{x}) = \varphi_{\alpha}(\mathbf{x})\mathbf{u}_{L}^{\alpha}(\mathbf{x}) \quad , \qquad (1)$$

where $\varphi_{\alpha}(\mathbf{x})$ is the partition of unity defined at a global node α. The partition of unity framework allows the exact reproduction of $\mathbf{u}_{L}^{\alpha}(\mathbf{x})$ in the global solution space. In the case of three-dimensional elasticity problems, local solution enrichments add only three degrees of freedom to the nodes of the coarse global mesh. Thus, the number of enrichment functions per global node does not depend on the number of degrees of freedom of the local problems. For further details of the method, refer to Duarte and Kim (2008) and Kim et al. (2010).

3. GENERALIZED FINITE ELEMENT SOLUTION USING THE PRECONDITIONED CONJUGATE GRADIENT METHOD

The conjugate gradient method is one of the most effective and simple methods which can solve a linear system of equations in an iterative fashion. It was first suggested by Hestenes and Stiefel (Hestenes and Stiefel 1952) and is based on the idea that the solution of the linear system of equations $\mathbf{Ax} = \mathbf{b}$ is the same as the minimization of potential $\Pi = \mathbf{x}'\mathbf{Ax}/2 - \mathbf{x}'\mathbf{b}$. The conjugate gradient method is, in fact, a semi-iterative type method because it is guaranteed in exact arithmetic to obtain the exact solution vector within *n* steps for a system of *n* linear equations. However, the convergence rate of the CGM greatly depends on the condition number of $\mathbf{A}$. Consequently, in practice, a preconditioner is almost always used with the conjugate gradient method to improve the conditioning of the matrix.

In this work, the stiffness matrix of the coarse global problem, which is solved to provide boundary conditions to local problems in *GFEM*^{gl}, is used as a pre-conditioner for the solution of the global problem enriched with the local solutions. The solution of the coarse global problem can be used as an initial guess for the solution of this enriched global problem. Although the numbers of DOFs of the coarse and enriched global problems do not match exactly, the preconditioner and initial guess with the same number of DOFs as that of the enriched global problem can be easily constructed with a small addition of computational cost. The solution procedure of the based on the PCGM scheme discussed above can be summarized as follows:

- Step 1. Solve a coarse global problem to provide the boundary conditions for local problems. This problem is solved by using a direct solver, and the factorized form of the stiffness matrix is preserved to use it as a preconditioner in Step 3.
- Step 2. Solve all local problems by using the solution of the coarse global problem as their boundary conditions. These local problems are also solved by the direct solver.
- Step 3. Select several global nodes and enrich them with the corresponding local solutions obtained from the previous step. This problem is solved by the PCGM, and the factorized form of the coarse global matrix obtained from the first step is used as a preconditioner. The initial guess required for the PCGM analysis at this step is the solution of the coarse global problem.

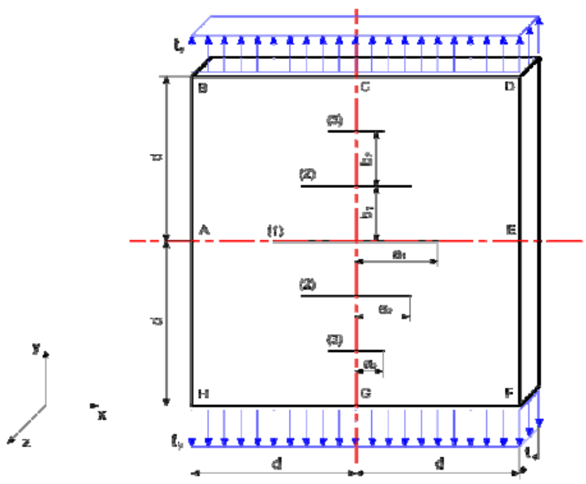

Fig. 1 Model problem description

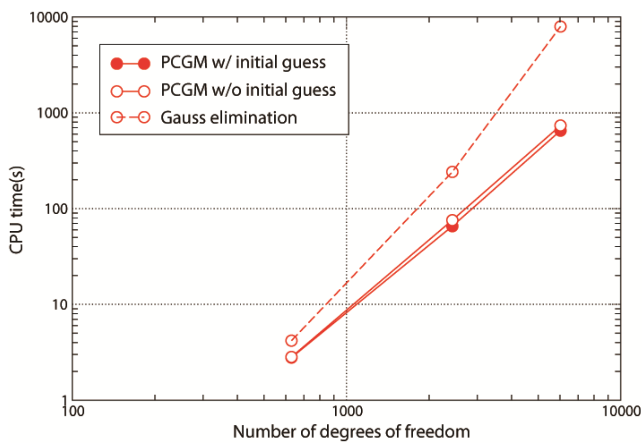

Fig. 2 Effect of problem size on CPU time

4. NUMERICAL EXPERIMENTS

As a preliminary investigation on the performance of the proposed approach, the effect of the size of the problem on its computational cost is examined. The model problem for this analysis is illustrated in Fig. 1, and it is a plate with five cracks at different locations. Only a half of the domain is modeled with respect to the vertical axis of symmetry indicated in the figure. It is based on the uniform mesh which has 6 * (6 × 12 × 1) = 432 tetrahedral elements. The following parameters are assumed in this model: Poisson's ratio = 0.33; Young's modulus E = 2.0×10^5; In-plane dimensions d = 36.0, a_1 = 18.0, a_2 = 12.0, a_3 = 6.0, b_1 = b_2 = 6.0; Domain thickness t = 6.0; Neumann boundary conditions t_y = 20.0. This problem is analyzed by using three different orders of polynomial enrichment, which are p = (1, 2, 3), in the domain. As the polynomial order of enrichment functions increases from one to three, the number of DOFs of the enriched global problem also increases and the corresponding numbers of DOFs are 630, 2430 and 6030, respectively. This problem is solved by using three different problem solving methods such as the PCGM with the coarse global solution as the initial guess, PCGM with a zero vector as the initial guess and Gauss elimination (LU factorization without pivoting). The tolerance level for convergence is defined by the Euclidean norm of the relative residual ($\|\mathbf{r}_{rel}\| = \|\mathbf{b} - \mathbf{Ax}\| / \|\mathbf{b}\|$) and set to 10^{-5} in this analysis.

Fig. 2 shows the CPU times required to solve the enriched global problem with the three different approaches described above. In the logarithmic scale plot of Fig. 2, it can be clearly seen that the slope of the curve is higher in the result obtained by the Gauss elimination than that by the two types of PCGM, which demonstrates the effectiveness of the proposed PCGM scheme. This coincides with the well-known fact that the number of operations in the direct solver is proportional to the cube of number of equations (n) while it is proportional to the square of number of equations in iterative methods. (Heath 2002) It can be also noticed that the PCGM with the coarse global solution as the initial guess requires slightly less CPU time than the one without it. This indicates that the coarse global solution is an initial guess with good quality and helps to improve the convergence of the enriched global problem.

5. CONCLUSIONS

In this paper, a new generalized finite element analysis using the PCGM analysis was introduced. It was shown that a preconditioner and initial guess of good quality can be constructed with a small addition of computational cost during the solution procedure of the $GFEM^{gl}$. A simple fracture mechanics example was analyzed to demonstrate the effectiveness of the proposed method. The results of this analysis revealed that the proposed method is especially efficient for problems of large size. The more attractive applications of the proposed approach such as multiple site damage (MSD) problems and crack propagation analysis are currently being explored.

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