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Sensor placement method for damage identification

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ABSTRACT

The sensitivity method in time domain has been applied extensively for damage identification. In this paper, the relationship between the error of damage identification and the sensitivity matrix is investigated based on the perturbation analysis. An index is defined based on the perturbation amplify effect and an optimal sensor placement method is proposed based on the minimization of that index. A sequential sub-optimal algorithm is presented which results in consistently good location selection. Numerical simulations with a two-dimensional high truss structure are conducted to validate the proposed method. Results reveal that the damage identification using the optimal sensor placement determined by the proposed method can identify multiple damages of the structure more accurately.

1. INTRODUCTION

Vibration damage detection methods have been developed for decades (Doebling 1996). The fundamental principle upon which vibration-based methods are founded is that the structural parameters are functions of the physical properties of the structure. A change in the physical properties is associated with some changes in the vibration responses which may be detected. Existing damage identification approaches can be divided into two categories of frequency domain methods and time domain methods. Natural frequencies, mode shapes, and their derivatives, such as the mode shape curvature (Padey 1991), flexibility matrices (Padey 1994), modal strain energy (Shi 2000b) etc., are usually taken as the measured information to identify the local structural damages in the frequency domain. Methods in time domain (Nagarajaiah 2009) include the Ibrahim time domain method, least-squares complex exponential method, the ERA methods and so on. With measurements of the structure in time domain, the location and severity of the local structural damage can be detected (Agbabian 1991).

More recently, Law and his colleagues develop the sensitivity-based damage identification approach based on the response sensitivity (Lu 2007a). Later, this method is developed for identifying both the system parameters and input excitation forces of a structure (Lu 2007b). It is found that the identification method based on the dynamic response sensitivity can provide more identification equations and only a few sensors are required. An adaptive regularization approach for solving the model

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updating problem is also presented (Li 2010). A changing limit on the summation of the identified parameters in each iteration step is applied to ensure that the physical significance of the structural parameters is retained.

It has been realized that the accuracy of the damage identification analysis may vary significantly with different spatial location of the response measurements (Sanayei 1996). Therefore, to identify damage successfully, proper measurement selection needs to be chosen carefully before field testing and damage identification analysis. Many methods, such as Effective Independence (EFI) method (Kammer 1992), Kinetic Energy Optimization Technique (EOT) (Heo 1997), Effective Independence-Driving Point Residue (EFI-DPR) method (Meo 2005) etc. have been developed for the determination of sensor locations in modal testing and condition monitoring of structures. Measurement selection for damage identification has also been studied. Cobb (1997) proposed the optimal sensor placement for structural damage detection by maximizing system observability. Shi (2000a) presented a method of optimizing sensor locations and detecting damage in a structure using the collected information. The sensor locations are prioritized according to their ability to localize structural damage based on the eigenvector sensitivity method. Only a small subset of the total structural degrees-of-freedoms (DOFs) is instrumented, and the in-completed modes yielded from these optimized sensor locations are used to localize structural damage. Xia (2000) proposed a concept of damage measurability that integrates damage sensitivity and noise sensitivity. Kripakaran (2009) used a global optimization approach to design the initial measurement locations for damage identification, and then, a greedy strategy is used to select measurement locations with maximum entropy among candidate model predictions.

In this paper, the relationship between the error and the sensitivity matrix in damage identification method based on dynamic response sensitivity in time domain is investigated based on the perturbation analysis. An index is defined based on the perturbation amplify effect and an optimal sensor placement method is proposed based on the minimization of that index. A sequential sub-optimal algorithm is presented which results in consistently good location selection. Numerical simulations with a two-dimensional high truss structure are conducted to validate the proposed method.

2. DAMAGE IDENTIFICATION METHOD BASED ON RESPONSE SENSITIVITY

2.1 Dynamic response of a structure

For a general finite element model of a time-invariant N degrees-of-freedom (DOFs) damped structure, the equation of motion can be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{L}\mathbf{F} \tag{1}$$

where M, C, and K are the mass, damping and stiffness matrices of the structural system respectively. Rayleigh damping is adopted which is of the form

$$\mathbf{C} = a_1 \cdot \mathbf{M} + a_2 \cdot \mathbf{K} \tag{2}$$

where a_1 and a_2 are constants to be determined from the modal damping ratios of two modes. $\ddot{\mathbf{x}}$, $\dot{\mathbf{x}}$ and $\mathbf{x}$ are vectors of acceleration, velocity and displacement of the structural system respectively. $\mathbf{F}$ is the vector of external excitation forces with matrix $\mathbf{L}$ mapping these forces to the associated DOFs of the structure. If the external excitation forces and the finite element model of the structure are known, responses $\ddot{\mathbf{x}}$, $\dot{\mathbf{x}}$ and $\mathbf{x}$ in Eq.(1) can be solved using the step-by-step Newmark-β integration method.

2.2 Dynamic response sensitivity method

Damage in a structure can be defined in terms of a stiffness reduction factor. The change in the global stiffness matrix is $\Delta\mathbf{K} = \sum_{i=1}^{ne} \alpha_i \mathbf{K}_i$, in which $\mathbf{K}_i$ is the stiffness matrix of the *i*th element, *ne* is the number of elements in the structure, α_i is the stiffness reduction factor of the *i*th element. The stiffness matrix of the damaged structure then becomes $\mathbf{K} + \Delta\mathbf{K}$. The notations on the stiffness change in the following study are defined as follows: $\boldsymbol{\alpha}$ and $\boldsymbol{\Delta\alpha}$ are the total stiffness change and the increment of an iteration with the size of $ne \times 1$, respectively. The superscript k in $\boldsymbol{\alpha}^k$ and $\boldsymbol{\Delta\alpha}^k$ denotes results obtained for the *k*th iteration. $\boldsymbol{\alpha}^k$ and $\boldsymbol{\Delta\alpha}^k$ are both vectors with size $ne \times 1$. α_i and $\Delta\alpha_i$ are the *i*th element of the vector $\boldsymbol{\alpha}$ and $\boldsymbol{\Delta\alpha}$.

Differentiate Eq. (1) with respect to α_i, we have

$$\mathbf{M}\frac{\partial\ddot{\mathbf{x}}}{\partial\alpha_i} + \mathbf{C}\frac{\partial\dot{\mathbf{x}}}{\partial\alpha_i} + \mathbf{K}\frac{\partial\mathbf{x}}{\partial\alpha_i} = -\frac{\partial\mathbf{K}}{\partial\alpha_i}\mathbf{x} - a_2\frac{\partial\mathbf{K}}{\partial\alpha_i}\dot{\mathbf{x}} \tag{3}$$

where $\partial\ddot{\mathbf{x}}/\partial\alpha_i$, $\partial\dot{\mathbf{x}}/\partial\alpha_i$, $\partial\mathbf{x}/\partial\alpha_i$ are vectors of the acceleration, velocity and displacement sensitivities with respect to the stiffness fractional change respectively. Since $\mathbf{x}$ and $\dot{\mathbf{x}}$ have been obtained from Eq. (1), the right-hand side of Eq. (3) can be considered as an equivalent forcing function, and Eq. (3) is of the same form as Eq. (1). The sensitivities $\partial\ddot{\mathbf{x}}/\partial\alpha_i$, $\partial\dot{\mathbf{x}}/\partial\alpha_i$ and $\partial\mathbf{x}/\partial\alpha_i$ can be also obtained by step-by-step Newmark-β integration method.

In the forward analysis, the dynamic responses and their sensitivities with respect to the structural parameters of a finite element system can be obtained from Eq. (1) and Eq. (3). In the inverse identification problem, the stiffness fractional change will be identified from the measured responses at the accessible DOFs. The most commonly used measured response is acceleration because of its ease of measurement.

An error function, defined as the difference between the calculated responses from the updated finite element model and the measured acceleration responses of the structure, can be written as

$$\Delta\ddot{\mathbf{x}} = \ddot{\mathbf{x}}_{\mathrm{m}} - \ddot{\mathbf{x}}_{\mathrm{cal}} \tag{4}$$

The identification equation can be expressed as the first order Taylor expansion of the acceleration responses (Lu 2007a)

$$\mathbf{S} \cdot \alpha = \Delta \ddot{\mathbf{x}} \tag{5}$$

where $\mathbf{S}$ is the acceleration response sensitivity matrix that can be calculated from Eq.(3), and it can be written as (Lu 2007a)

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \cdots \\ \mathbf{S}_j \\ \cdots \\ \mathbf{S}_{nt} \end{bmatrix} \quad with \;\; \mathbf{S}_i = \begin{bmatrix} \frac{\partial \ddot{\mathbf{x}}_1(t_j)}{\partial \alpha_1} & \frac{\partial \ddot{\mathbf{x}}_1(t_j)}{\partial \alpha_2} & \frac{\partial \ddot{\mathbf{x}}_1(t_j)}{\partial \alpha_3} & \cdots & \frac{\partial \ddot{\mathbf{x}}_1(t_j)}{\partial \alpha_{ne}} \\ \frac{\partial \ddot{\mathbf{x}}_2(t_j)}{\partial \alpha_1} & \frac{\partial \ddot{\mathbf{x}}_2(t_j)}{\partial \alpha_2} & \frac{\partial \ddot{\mathbf{x}}_2(t_j)}{\partial \alpha_3} & \cdots & \frac{\partial \ddot{\mathbf{x}}_2(t_j)}{\partial \alpha_{ne}} \\ \frac{\partial \ddot{\mathbf{x}}_3(t_j)}{\partial \alpha_1} & \frac{\partial \ddot{\mathbf{x}}_3(t_j)}{\partial \alpha_2} & \frac{\partial \ddot{\mathbf{x}}_3(t_j)}{\partial \alpha_3} & \cdots & \frac{\partial \ddot{\mathbf{x}}_3(t_j)}{\partial \alpha_{ne}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \ddot{\mathbf{x}}_{N_m}(t_j)}{\partial \alpha_1} & \frac{\partial \ddot{\mathbf{x}}_{N_m}(t_j)}{\partial \alpha_2} & \frac{\partial \ddot{\mathbf{x}}_{N_m}(t_j)}{\partial \alpha_3} & \cdots & \frac{\partial \ddot{\mathbf{x}}_{N_m}(t_j)}{\partial \alpha_{ne}} \end{bmatrix} \tag{6}$$

where nt is the total number of time sampling points, N_m is the number of sensors and *ne* is the number of structural parameters to be identified. $\mathbf{S} \in \Re^{(N_m \times nt) \times ne}$ is a two dimensional matrix representing three-dimensional array, with one dimension of time, one dimension of measured DOFs and the other dimension of the number of structural parameters to be identified.

The relationship between $\Delta \ddot{\mathbf{x}}$ and $\mathbf{S}$ is nonlinear. Eq.(5) omits the high order terms, thus, a nonlinear model updating technique, like the Gauss-Newton method, is required. An analytical model of the target structure is treated as the reference model, and the measurements from the damaged structure will be used to update the reference model with iterations. The vector of structural physical parameters can be identified through model updating. The damage identification equation for the (*k*+1)th iteration can be written as

$$\mathbf{S}^k \boldsymbol{\Delta\alpha}^{k+1} = \boldsymbol{\Delta}\ddot{\mathbf{x}}^k \tag{7}$$

where $\mathbf{S}^k$ and $\boldsymbol{\Delta}\ddot{\mathbf{x}}^k$ are obtained from the *k*th iteration. Convergence is considered achieved when the following criterion is met

$$\frac{\left\| \boldsymbol{\Delta\alpha}^{k+1} - \boldsymbol{\Delta\alpha}^{k} \right\|}{\left\| \boldsymbol{\Delta\alpha}^{k+1} \right\|} \le Tolerance \tag{8}$$

where $\boldsymbol{\Delta\alpha}^k$ is the stiffness fractional change from the *k*th iterative step. *Tolerance* is a small value to be defined. The final fractional change in the stiffness of the *i*th element

after the *n*th iteration is

$$\alpha_i = \alpha_i^0 + \Delta\alpha_i^1 + \Delta\alpha_i^2 + \cdots + \Delta\alpha_i^n \tag{9}$$

in which *n* is the number of iterations, α_i^0 is the assumed as the initial stiffness reduction factor that is usually assumed equal to 0.

2.3 Adaptive Tikhonov regularization method

Like many other inverse problems, the solution of Eq. (7) is often ill-conditioned and regularization techniques are needed to provide bounds to the solution. The aim of regularization in the inverse analysis is to promote certain regions of parameter space where the model realization should exist. The two most widely used regularization methods are Tikhonov regularization (Tikhonov 1995) and truncated singular value decomposition(Hansen 1987). In Tikhonov regularization, Eq. (7) can be redefined as the minimization of the following objective function

$$J(\Delta\boldsymbol{\alpha}^{k+1}) = \left\|\mathbf{S}^k\Delta\boldsymbol{\alpha}^{k+1} - \Delta\ddot{\mathbf{x}}^k\right\|_2^2 + \lambda^2\left\|\Delta\boldsymbol{\alpha}^{k+1}\right\|_2^2 \tag{10}$$

This function has two parts: the first part minimizes the difference between measured and calculated quantities; the second part restricts the size of the update solutions. The regularization parameter λ controls the weight given to the solution norm $\left\|\Delta\boldsymbol{\alpha}^{k+1}\right\|_2$ relative to the residual norm $\left\|\mathbf{S}^k\Delta\boldsymbol{\alpha}^{k+1} - \Delta\ddot{\mathbf{x}}^k\right\|_2$. The key point of regularization is to find an optimal λ, and then the identification equation can be solved. The L-curve method (Hansen 1992) is commonly used to find the optimal regularization parameter λ.

Inverse problem is always ill-posed and measurement noise may have adverse effect in the process of identification. The iterative identification methods should be able to ensure the significance of the structural parameters and mitigate the unfavorable effect of noise in identification. An adaptive regularization method with an adaptive upper limit on the identified damage (Li 2010) based on results from last iteration steps is adopted. It has been shown that the adaptive Tikhonov regularization has obvious advantage over the traditional Tikhonov regularization with less false positives and false negatives especially when relatively high noise level exists in the measurements. The objective function of the optimization is expressed as

$$J(\Delta\boldsymbol{\alpha}^{k+1}) = \left\|\mathbf{S}^k\Delta\boldsymbol{\alpha}^{k+1} - \Delta\ddot{\mathbf{x}}^k\right\|_2^2 + \lambda^2\left\|\sum_{l=1}^{k+1}\Delta\boldsymbol{\alpha}^l - \boldsymbol{\alpha}^{k,*}\right\|_2^2 \tag{11}$$

where $\boldsymbol{\alpha}^{k,*}$ is a value to coordinate the constraint of the solution in the *k*th iteration in the model updating process. Parameter $\boldsymbol{\alpha}^{k,*}$ is defined as

$$\left(\boldsymbol{\alpha}^{k,*}\right)_i = \begin{cases} 0 & if \ \left(\sum_{l=1}^{k} \boldsymbol{\Delta\alpha}^l\right)_i \geq 0 \\ \left(\sum_{l=1}^{k} \boldsymbol{\Delta\alpha}^l\right)_i & if \ \left(\sum_{l=1}^{k} \boldsymbol{\Delta\alpha}^l\right)_i < 0 \end{cases} \quad \left(i = 1, 2, \cdots, ne\right) \tag{12}$$

The adaptive Tikhonov regularization method has been proposed to improve the damage identification results by separating all the structural elements to be assessed into two categories of possible damaged elements and intact elements from results obtained in the previous iteration. The perturbations of elemental stiffness reduction factors of the possible damaged elements in each iteration are then limited to the cumulative identified change of stiffness $\left(\sum_{l=1}^{k} \boldsymbol{\Delta\alpha}^l\right)_i$ and the reduction factors of other elements are restrained close to zeros.

To minimize the objective function in Eq. (11), the sensitivity matrix is singular value decomposed as

$$\mathbf{S}^k = \mathbf{U} \sum \mathbf{V}^{\mathbf{T}} \tag{13}$$

where $\mathbf{U}$ and $\mathbf{V}$ are of dimensions $\left(N_m \times nt\right) \times ne$ and $ne \times ne$ respectively with $\left(N_m \times nt\right) \geq ne$ .The vectors in $\mathbf{U}$ and $\mathbf{V}$ are orthogonal, i.e., $\mathbf{U}^{\mathbf{T}}\mathbf{U} = \mathbf{V}^{\mathbf{T}}\mathbf{V} = \mathbf{V}\mathbf{V}^{\mathbf{T}} = \mathbf{I}$. However, $\mathbf{U}\mathbf{U}^{\mathbf{T}} \neq \mathbf{I}$, because matrix $\mathbf{U}$ contains only ne columns in the thin version of the singular value decomposition. This thin version decomposition is more economical and is usually sufficient for calculation (Weber 2009). $\sum = diag\left(\sigma_1, \sigma_2, \cdots, \sigma_{ne}\right)$ has the singular values σ_i arranged in a non-increasing order such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{ne} \geq 0$.

The solution to Eq. (11) can then be obtained as a function of the regularization parameter λ (Wang 2012),

$$\Delta\boldsymbol{\alpha}^{k+1} = (\mathbf{S}_k{}^{\mathbf{T}}\mathbf{S}_k + \lambda^2\mathbf{I})^{-1}[\mathbf{S}_k{}^{\mathbf{T}}\Delta\ddot{\mathbf{x}}^k - \lambda^2(\boldsymbol{\alpha}^k - \boldsymbol{\alpha}^{k,*})] = \sum_{i=1}^{ne}\left(\mathbf{V}_i f_i \mathbf{x}_w - V_i(1 - f_i)\mathbf{x}_v\right) \tag{14}$$

where $\mathbf{x}_w = \dfrac{\mathbf{U}_i{}^{\mathrm{T}}\Delta\ddot{\mathbf{x}}^k}{\sigma_i}$, $x_v = \mathbf{V}^{\mathrm{T}}\left(\alpha^k - \alpha^{k,*}\right)$ and $f_i = \dfrac{\sigma_i^2}{\left(\sigma_i^2 + \lambda^2\right)}$ is called filter factor. Since the range of the regularization parameter is $\sigma_1 \geq \lambda \geq \sigma_{ne}$. It is noted that regularization method makes use of the filter factors to damp the effects associated with small singular values.

3. OPTIMAL SENSOR PLACEMENT METHODOLOGY

The adaptive Thikhonov regularization method is applied to solve the ill-conditioning identification equation. However, the identification results can be significantly influenced with the spatial location of the response measurements. The identification results will be bad with non-proper sensor placement although adaptive Thikhonov regularization is carried out.

The relationship between the identification error and the sensitivity matrix is investigated and an optimal sensor placement method is proposed in the following paragraphs.

3.1 Perturbation analysis

The identification equation for the first iteration is that

$$\mathbf{S}^0 \boldsymbol{\Delta\alpha}^1 = \boldsymbol{\Delta}\ddot{\mathbf{x}}^0 \tag{15}$$

in which $\mathbf{S}^0$ is the sensitivity matrix calculated from the initial state of the structural reference model. For ease of discussion, it is assumed that Eq.(15) is a determined linear equation (the number of equation equals to the number of unknowns) and matrix $\mathbf{S}^0$ is non-singular (with full rank). In this case, Eq.(15) has a unique least-square solution which is

$$\boldsymbol{\Delta\alpha}^1 = \left(\mathbf{S}^0\right)^{-1} \boldsymbol{\Delta}\ddot{\mathbf{x}}^0 \tag{16}$$

Assume the perturbations in structural model and measurements are independent. The influences of these two kinds of perturbations on identification results are investigated respectively.

3.1.1 Perturbation analysis on measurement noise

Assume the noise in measurement $\ddot{\mathbf{x}}_{\mathrm{m}}$ is $\varepsilon_{\ddot{x}}$ such that the perturbation in right term of Eq. (15) is $\varepsilon_{\ddot{x}}$ and it will result in perturbation $\varepsilon_{\alpha n}$ in $\boldsymbol{\Delta\alpha}^1$. The identification equation with noise perturbation can be written as

$$\mathbf{S}^0 \left(\boldsymbol{\Delta\alpha}^1 + \varepsilon_{\alpha n}\right) = \boldsymbol{\Delta}\ddot{\mathbf{x}}^0 + \varepsilon_{\ddot{x}} \tag{17}$$

$\varepsilon_{\alpha n}$ is given by,

$$\varepsilon_{\alpha n} = \left(\mathbf{S}^0\right)^{-1} \varepsilon_{\ddot{x}} \tag{18}$$

According to the compatibility of matrix norm and vector and the theorem of triangle inequality (Golub 1996), we have

$$\begin{cases} \left\|\boldsymbol{\Delta}\ddot{\mathbf{x}}^0\right\| \le \left\|\mathbf{S}^0\right\| \left\|\boldsymbol{\Delta\alpha}^1\right\| \\ \left\|\varepsilon_{\alpha n}\right\| \le \left\|\left(\mathbf{S}^0\right)^{-1}\right\| \left\|\varepsilon_{\ddot{x}}\right\| \end{cases} \tag{19}$$

Then we have

$$\frac{\|\varepsilon_{\alpha n}\|}{\|\Delta\boldsymbol{\alpha}^1\|} \le \|\mathbf{S}^0\|\left\|\left(\mathbf{S}^0\right)^{-1}\right\|\frac{\|\varepsilon_{\ddot{x}}\|}{\|\Delta\ddot{\mathbf{x}}^0\|} \tag{20}$$

It is founded that the perturbation $\varepsilon_{\alpha n}$ in $\Delta\boldsymbol{\alpha}^1$ will be amplified $\|\mathbf{S}^0\|\left\|\left(\mathbf{S}^0\right)^{-1}\right\|$ times comparing with the perturbation $\varepsilon_{\ddot{x}}$ in $\Delta\ddot{\mathbf{x}}^0$. In matrix analysis, $\|\mathbf{S}^0\|\left\|\left(\mathbf{S}^0\right)^{-1}\right\|$ is defined as the condition number of $\mathbf{S}^0$

$$cond(\mathbf{S}^0) = \|\mathbf{S}^0\|\left\|\left(\mathbf{S}^0\right)^{-1}\right\| \tag{21}$$

3.1.2 Perturbation analysis on model errors

Assume ε_s as the perturbation in $\mathbf{S}^0$ caused by structural model errors and it will result in perturbation $\varepsilon_{\alpha m}$ in $\Delta\boldsymbol{\alpha}^1$, that is

$$\left(\mathbf{S}^0 + \varepsilon_s\right)\left(\Delta\boldsymbol{\alpha}^1 + \varepsilon_{\alpha m}\right) = \Delta\ddot{\mathbf{x}}^0 \tag{22}$$

Then we have,

$$-\varepsilon_{\alpha m} = \left(\mathbf{S}^0\right)^{-1}\varepsilon_s\Delta\boldsymbol{\alpha}^1 + \left(\mathbf{S}^0\right)^{-1}\varepsilon_s\varepsilon_{\alpha m} \tag{23}$$

According to the compatibility of matrix norm and vector and the theorem of triangle inequality (Golub 1996), it has

$$(1-\left\|\left(\mathbf{S}^0\right)^{-1}\right\|\|\varepsilon_s\|)\|\varepsilon_{\alpha m}\| \le \left\|\left(\mathbf{S}^0\right)^{-1}\right\|\|\varepsilon_s\|\|\Delta\boldsymbol{\alpha}^1\| \tag{24}$$

when $\left\|\left(\mathbf{S}^0\right)^{-1}\right\|\|\varepsilon_s\| < 1$,

$$\frac{\|\varepsilon_{\alpha m}\|}{\|\Delta\boldsymbol{\alpha}^1\|} \le \frac{\|\mathbf{S}^0\|\left\|\left(\mathbf{S}^0\right)^{-1}\right\|}{1-\|\varepsilon_s\|\left\|\left(\mathbf{S}^0\right)^{-1}\right\|}\frac{\|\varepsilon_s\|}{\|\mathbf{S}^0\|} \tag{25}$$

It is founded that the perturbation $\varepsilon_{\alpha m}$ in $\Delta\boldsymbol{\alpha}^1$ is also proportional to condition number of $\mathbf{S}^0$.

According to the perturbation analysis, it can be concluded that condition number of the sensitivity matrix $\mathbf{S}^0$ can be a measure of the perturbation amplify effect of the

identification equation. When the condition number of the sensitivity matrix is large, the measurement and structural model errors will propagate significant amplification.

The above discussion is on the assumption that Eq.(15) is a determined linear equation. However, in time domain method, Eq.(15) is always over-determined and it can be expressed as

$$\left(\mathbf{S}^0\right)^{\mathrm{T}} \mathbf{S}^0 \boldsymbol{\Delta\alpha}^1 = \left(\mathbf{S}^0\right)^{\mathrm{T}} \boldsymbol{\Delta}\ddot{\mathbf{x}}^0 \tag{26}$$

Since

$$cond\left(\left(\mathbf{S}^0\right)^{\mathrm{T}} \mathbf{S}^0\right) = \left[cond\left(\mathbf{S}^0\right)\right]^2 \tag{27}$$

It is noted that for the over-determined identification problem, $\left[cond\left(\mathbf{S}^0\right)\right]^2$ can measure the perturbation amplify effect of the over-determined identification equations and an index for the proposed sensor placement method is defined based on the perturbation amplify effect as

$$J = \left[cond\left(\mathbf{S}^0\right)\right]^2 \tag{28}$$

3.2 Optimal sensor placement method

The goal of the proposed sensor placement algorithm is to minimize the perturbation amplify effect index *J* in Eq. (28). It is noted that the proposed sensor placement method is derived from the undamaged structure and thus it is independent of the damage configuration. Suppose the maximum number of sensor is N_s and the number of sensor candidate is N_c. The total number of the possible sensor combinations N_{com} is given by the following binomial coefficient

$$N_{com} = C_{N_c}^{N_s} = \frac{N_s!}{\left(N_c - N_s\right)! N_s!} \tag{29}$$

It is obvious that for a structure finite element model with several thousands of DOFs, it is impossible to test all combinations, even for a few number of sensors. A sub-optimal sensor placement method is therefore required.

A sequential sensor placement algorithm is proposed in which sensors are computed sequentially by placing one sensor at a time in the structure at a position that results in the minimization of index *J*. The detail algorithm for the optimal sensor placement is as follows:

1. Initialize: no sensors are selected, number of sensor $N_m = 0$ and the optimal sensor placement combination is $L_0 = \{\ \}$.

2. If number of sensor N_m < maximum number of sensors N_s, do
   a. Consider combinations with one additional sensor, $N_m = N_m + 1$
   b. For counter $i = 1$ to number of possible sensor locations $\left(N_c - N_m + 1\right)$
      i. Obtain sensor configuration $L_{N_m}\left(i\right)$ by adding sensor *i* to configuration $L_{(N_m - 1)}$
      ii. Calculate the sensitivity matrix $S^0\left[L_{N_m}\left(i\right)\right]$ with the new sensor configuration L_{N_m}
      iii. Evaluate the index $J\left[L_{N_m}\left(i\right)\right]$.
   c. End
   d. Select the sensor configuration $L_{N_m}\left(i\right)$ that minimizes the index J as the updated sensor configuration L_{N_m}
   e. If $J\left(L_{N_m}\right) > J\left(L_{N_m - 1}\right)$, break
3. End

The algorithm can be stopped by two ways. The first way is that the criterion of $J\left(L_{N_m}\right) > J\left(L_{N_m - 1}\right)$ is satisfied, then the algorithm can be stopped and the sensor configuration $L_{(N_m - 1)}$ is selected as the optimal sensor placement. The second stop way is that when the number of sensor N_m equals to the maximum number of sensor N_s and the sensor configuration L_{N_m} is selected as the optimal sensor placement.

The algorithm for the worst sensor placement is also required for comparison. It is similar to the algorithm for the optimal sensor placement and the differences between them are the step 2.d and 2.e., In 2.d the sensor configuration that maximizes the index *J* will be chosen and in 2.e the stop criterion is changed to $J\left(L_{N_m}\right) < J\left(L_{N_m - 1}\right)$.

4. NUMERICAL STUDIES

4.1 Information of the two-dimensional high truss structure

A two-dimensional 50-meter high planar truss structure (Law 2011) shown in Fig.1 is investigated to illustrate the proposed sensor placement method. It is a simplified model of a popular type of power transmission tower structure in China. The truss structure consists of 14 nodes each with two DOFs. The two ends of each truss element are assumed hinged and the structure is found on supports at Nodes 1 and 2 with hinges. It has five levels with 10m high each. The mass density of material is 7.8×10^3 kg/m^3 and the elastic modulus of material is 2.06 GPa. The cross-sectional

area of each truss element is $2.5\times10^{-3}m^2$. Rayleigh damping is assumed and the damping ratios for the first two modes are both taken as $\xi = 0.02$.

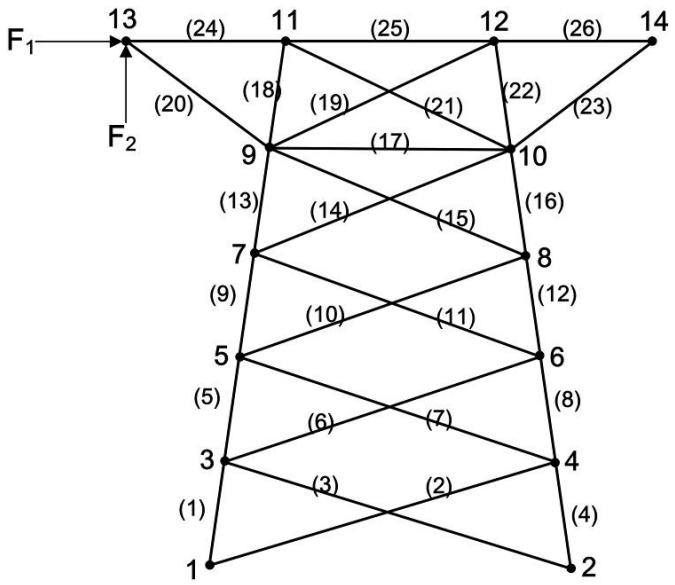


Fig. 1 Finite element model of the two-dimensional high truss structure (•) denotes the element number

Two multiple sine wave external excitations are applied along the horizontal and vertical directions at Node 13. These two forces are,

$$\begin{aligned}\mathbf{F}_1(t) &= 325\sin(10\pi t) + 200\sin(30\pi t + 0.5\pi) + 165\sin(80\pi t + 0.9\pi) \quad N \\ \mathbf{F}_2(t) &= 300\sin(20\pi t + 0.1\pi) + 160\sin(40\pi t + 0.4\pi) + 220\sin(100\pi t + 1.1\pi) \quad N\end{aligned} \tag{30}$$

The sampling rate is 500Hz and the total sampling points is 250. When there is noise in the “measured” response, the polluted response is simulated by adding a random component to the calculated responses as

$$\ddot{\mathbf{x}}_{\mathrm{m}} = \ddot{\mathbf{x}} + E_P \mathbf{N}_{noise} \sigma(\ddot{\mathbf{x}}) \tag{31}$$

where $\ddot{\mathbf{x}}_{\mathrm{m}}$ and $\ddot{\mathbf{x}}$ denote the polluted measured responses and the calculated responses without noise respectively. E_p is the percentage noise level, $\mathbf{N}_{noise}$ is a standard normal distribution vector with zero mean and unit standard deviation, $\sigma(\ddot{\mathbf{x}})$ is the standard deviation of the calculated acceleration response. In this simulation, measured responses with 5% measurement noise are considered. There is a total of 24 DOFs in the structure, and acceleration responses in the horizontal and vertical directions at Nodes 7, 8, 9, 10, 11, 12, 13 and 14 with $N_c = 16$ are assumed to be the

candidate sensor locations. The maximum number of sensor N_s is assumed as 6.

4.2 Analysis on index J

The optimal and worst sensor combinations are calculated by the proposed sensor placement method. The optimal sensor combination includes six sensor locations which are N11(y), N14(x), N7(y), N11(x), N8(y) and N12(y) and the worst sensor combination includes only two sensor locations which are N13(y) and N14(y). N14(x) and N11(y) denote the horizontal direction of Node 14 and vertical direction of Node 11 respectively. The perturbation amplify effect index *J* of the optimal sensor combination is 123.56 which is much smaller than that of the worst sensor combination 3.15×10^7.

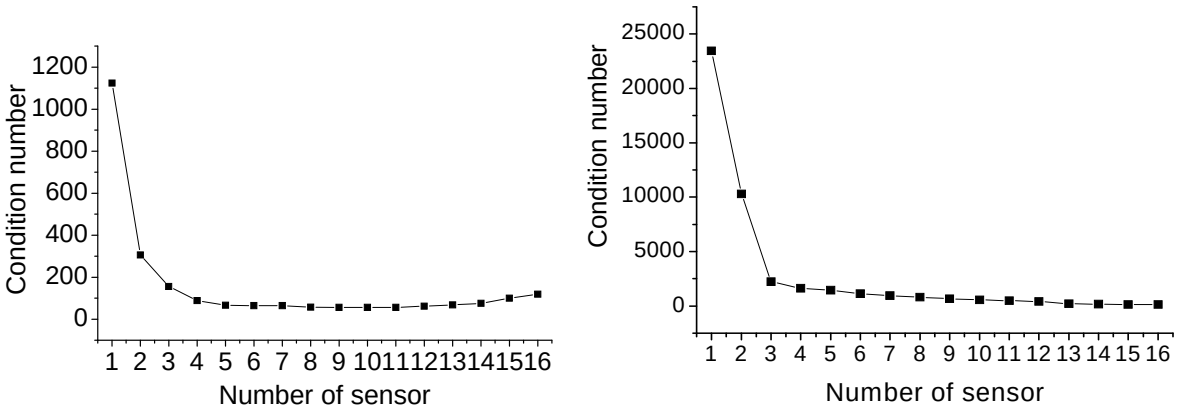
4.3 Analysis on the effect of number of sensor

To study the effect of number of sensor, the maximum number of sensor N_s is assumed equal to the candidate sensor location number, that is $N_s = N_c = 16$ for both of the optimal and worst sensor placement algorithms. The iterative procedures for both algorithms are allowed to repeat for the all of the 16 candidate sensor locations without stopping criterion. Table 1 lists the sequence of the selection of the two kinds of sensor configurations. Figures 2(a) and 2(b) show the relationship between the number of sensor and condition number of the sensitivity matrix calculated by the optimal and worst sequential sensor placement algorithms. The sensor placements corresponding

Table 1 Sensor location sequence for the best and worst sensor combinations

Sequence No.	Location	
	Optimal sensor combination	Worst sensor combination
1	N11(y)	N14(y)
2	N14(x)	N14(x)
3	N7(y)	N13(y)
4	N11(x)	N12(x)
5	N8(y)	N11(x)
6	N12(y)	N13(x)
7	N12(x)	N10(x)
8	N7(x)	N9(x)
9	N9(y)	N10(y)
10	N10(y)	N12(y)
11	N13(x)	N9(y)
12	N10(x)	N11(y)
13	N9(x)	N8(y)
14	N8(x)	N7(y)
15	N14(y)	N8(x)
16	N13(y)	N7(x)

to different number of sensor are sequentially selected according to the sequences listed in Table 1. The smallest condition number corresponding to the optimal sensor placement is 55.88, with 11 sensors while the biggest condition number is 1123.96, with only one sensor. The smallest condition number corresponding to the worst sensor placement is 119.16, with 16 sensors while the biggest condition number is 2.34×10^4 with one sensor. The condition number with the worst sensor combination is much bigger than that with the most optimal sensor combination.



(a) Optimal sensor combination (b) Worst sensor combination

Fig. 2 Relationship between the number of sensor and condition number of the sensitivity matrix

4.4 Discussion on identification results

To illustrate the advantage of the measurement selection based on the presented sensor placement method in structural damage identification, two damaged states each with three different sensor placements are considered. The detail information is listed in Table 2. In the first damage state, element 16 is assumed deteriorated with a stiffness reduction by 10% while in the second state, damages are assumed on elements 4 and 13, each with 10% stiffness reduction. Averaging of 50 simulations of response reconstruction are carried out for each case.

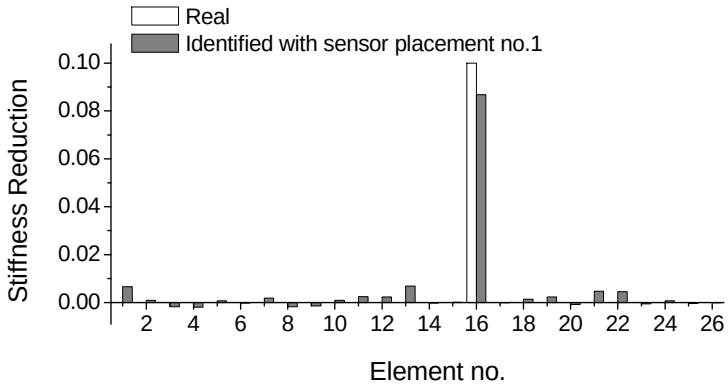
Figures 3(a), 3(b) and 3(c) are the identification results for the case 1, 2 and 3 respectively. It is founded that the identification results with sensor placement no.1 (optimal sensor placement with 6 sensors) are of great accuracy. For the identification results with sensor placement no.2 (worst sensor placement with 2 sensors), element 13 is wrongly identified as damaged element with more or less -3% damage ratio. The location of the real damaged element 16 can be identified but the damage ratio is much smaller than the true value. The identification results with sensor placement no.3 (worst sensor placement with 6 sensors) are similar to those with sensor placement no.2.

Figures 4(a), 4(b) and 4(c) are the identification results for the case 4, 5 and 6 respectively. It is founded that the identification results with sensor placement no.1 are still of great accuracy. For sensor placement no.2, elements 1 and 16 are wrongly detected as damaged elements. The locations of the real damaged elements 4 and 13

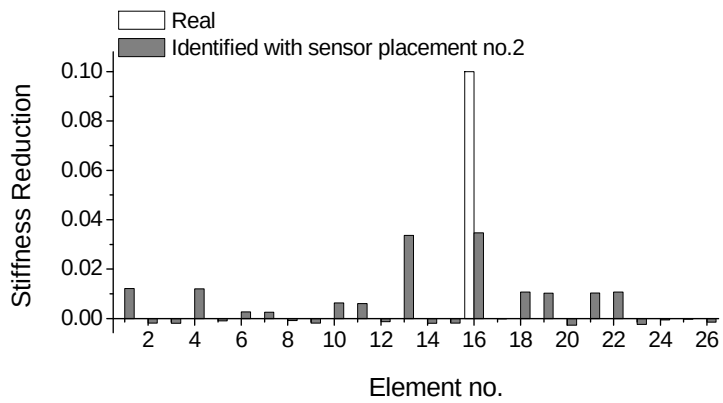
can be identified. The identified damage ratio of element 13 can be accepted while that of the element 4 is much smaller than the true value. Sensor placement no.4 is obtained from the optimal sensor location algorithm, and it has only two sensors. The location and damage ratio of the damage state 2 with multi-damages can be accurately identified as well.

Table 2 Detail of damage cases and sensor combinations

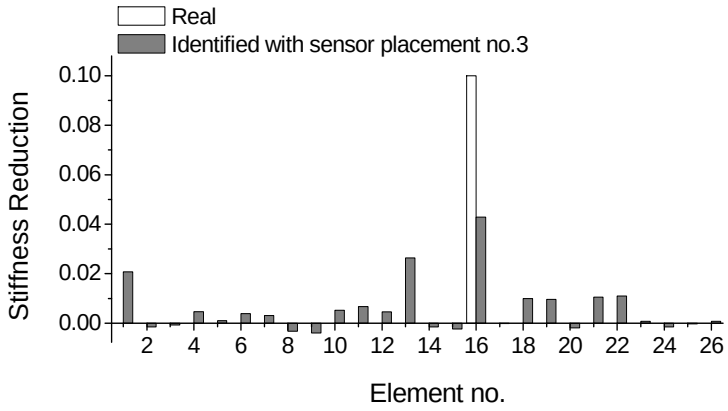
Case	Damage state	Damaged elements (damage rations)	Sensor placement no. (Type, sensor number)	Sensor locations
1	1	Element 16, (-10%)	1(Optimal, 6)	N11(y), N14(x) N7(y), N11(x) N8(y), N12(y)
2			2(Worst, 2)	N14(y), N14(x)
3			3(Worst, 6)	N14(y), N14(x) N13(y), N12(x) N11(x), N13(x)
4	2	Element 4, (-10%) Element 13, (-10%)	1(Optimal, 6)	N11(y), N14(x) N7(y), N11(x) N8(y), N12(y)
5			2(Worst, 2)	N14(y), N14(x)
6			4(Optimal, 2)	N11(y), N14(x)



(a) Case 1

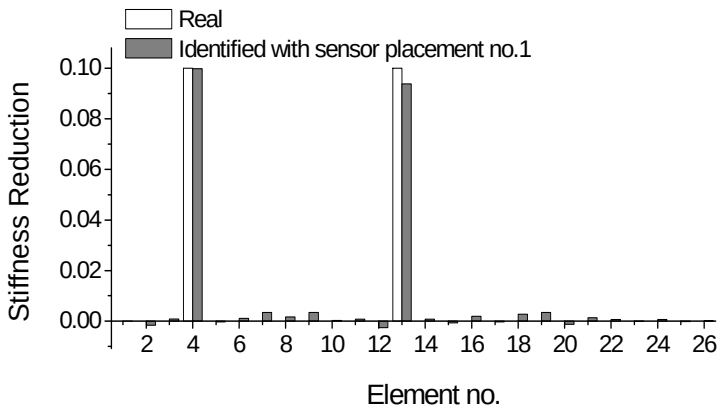


(b) Case 2

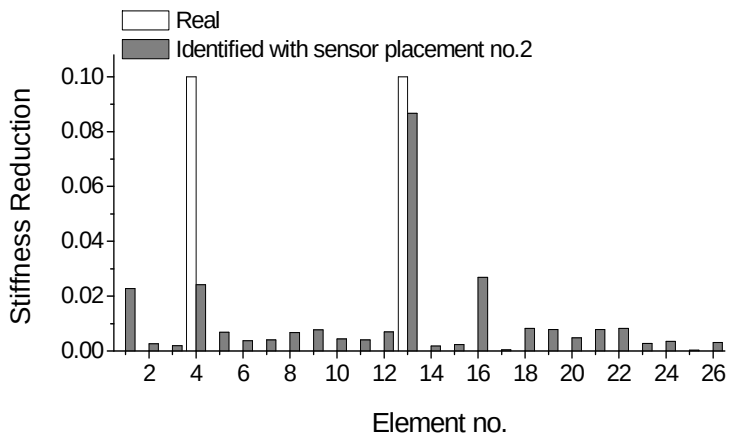


(c) Case 3

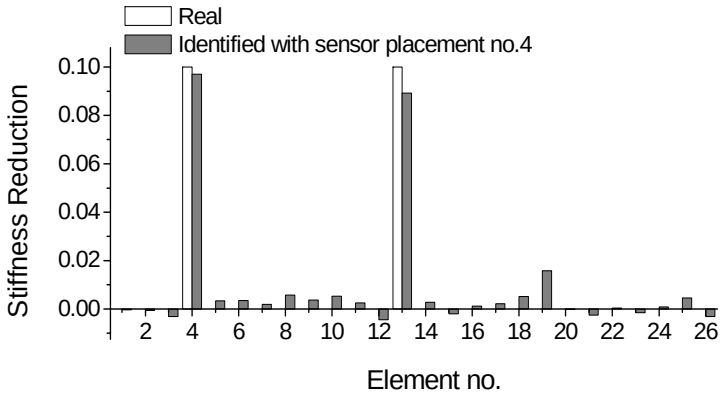
Fig. 3 - Identification result of damage state 1 (for the truss structure)



(a) Case 4



(b) Case 5



(c) Case 6

Fig. 4 - Identification result of damage state 2 (for the truss structure)

From the above observations, it can be concluded that, with the sensor placement derived by the proposed optimal sensor placement algorithm, the damage locations and damage ratios can be identified accurately while those sensor placements derived by the worst sensor placement algorithm will cause false detection results.

5. CONCLUSIONS

In this paper, the relationship between the error and the sensitivity matrix in damage identification method based on dynamic response sensitivity in time domain is investigated based on the perturbation analysis. An index is defined based on the perturbation amplify effect and an optimal sensor placement method is proposed based on the minimization of that index. A sequential sub-optimal algorithm is presented which results in consistently good location selection. The proposed sensor placement method is derived from the undamaged structure and thus it is independent of the damage configuration. Numerical simulations with a two-dimensional high truss structure are conducted to validate the proposed method. Results reveal that the damage identification using the optimal sensor placement determined by the proposed method can identify multiple damages of the structure more accurately.

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