The 2013 World Congress on
Advances in Structural Engineering and Mechanics (ASEM13)
Jeju, Korea, September 8-12, 2013

A novel algorithm for solving MDOF systems with clearances

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ABSTRACT

The paper presents a novel algorithm for solving multi-degree-of-freedom (MDOF) systems containing clearance type nonlinearities. The algorithm is based on piecing together the local linear solutions. The accuracy of the solution is the same at any time point inside the response since the local solutions are obtained analytically. The time step should be only sufficiently small for a reliable numerical determination of the switching point between two local solutions.

Considering the proposed algorithm, the nonlinear frequency response of the three-degree-of-freedom semi-definite system with two clearances are analyzed. Good correlations with the results obtained by the numerical integration confirm the algorithm.

1. INTRODUCTION

Demands for utilizing nonlinear structural components are increasingly present in modern engineering applications. Clearance type nonlinearities can be often found in power transmission systems that contain components such as gears, bearings, clutches, in robot joints and guideways of mechatronic systems, in many demountable structures as a result of looseness of joints, etc. Clearances in transmission systems do not only exist due to manufacturing errors and failures, they are often involved by their function (e.g. geared systems require clearance between mating gears for smooth operation).

Numerous methods, such as standard time-domain numerical integrations, various harmonic balance methods (Chatterjee 1996, Kahraman 1990, Kim 2005, Lau 1983, Wong 1991), analog and digital simulations, piecewise linear techniques (Thompson 1986) and the time finite element method (Kranjcevic 2001, Kranjcevic 2007, Wang 1995) have been developed because of the highly individualistic nature of nonlinear systems.

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In this paper, a novel algorithm for calculating steady state responses of multi-degree-of-freedom (MDOF) systems with clearances is presented. The algorithm is a new developed semi-analytical procedure of explicit integration based on piecing together the local linear equations of motion (Kranjcevic 2003). The system with clearances starts from an initial position described with one of linear equations of motion. When the system changes the piecewise linear stiffness region, the motion is represented with the new linear equation. The determination of times in which the system changes a linear equation of motion can be done only numerically.

The accuracy of the method does not significant depend on a magnitude of the integration step since the local linear equations of motion are being solved analytically. The time step has to be sufficiently small for a reliable numerical determination of the switching points between linear equations of motion. It is a remarkable advantage with respect to other numerical integration methods

The method is applied to obtain the frequency response of the three-degree-of-freedom semi-definite system with two clearances under periodic excitations. The numerical results are validated considering standard numerical integration software (MATLAB).

2. PROBLEM FORMULATION

A multi-degree-of-freedom system (MDOF) with clearances can be represented as the n+1 degree-of-freedom semi-definite model shown in Fig. 1. The model consists of n+1 mass elements and n clearance type nonlinearities given by the function $H(\bar{x}_i, \dot{\bar{x}}_i)$, $i = 1(1)\mathrm{n}$ where $\bar{x}_i$ and $\dot{\bar{x}}_i$ are the relative displacement and velocity. The system is excited by the periodic force $\boldsymbol{f}(t)$.

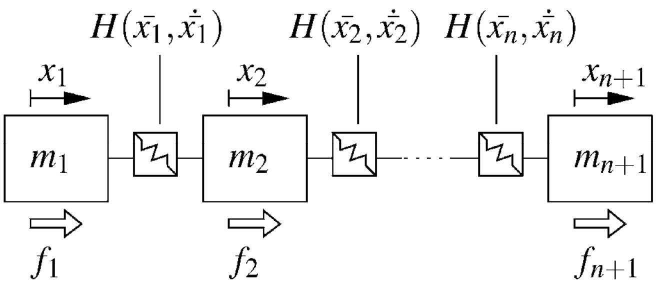

Fig. 1 n+1-degree-of-freedom semi-definite system with clearances

With $\boldsymbol{x} = [x_1 \quad x_2 \quad \cdots \quad x_{\mathrm{n+1}}]^{\mathrm{T}}$ and $\bar{\boldsymbol{x}} = \left[(x_1 - x_2) \quad \cdots \quad (x_{\mathrm{n}} - x_{\mathrm{n+1}})\right]^{\mathrm{T}} = \left[\bar{x}_1 \quad \cdots \quad \bar{x}_{\mathrm{n}}\right]^{\mathrm{T}}$ being the absolute and relative displacements, the equation of motion is expressed as

$$\boldsymbol{M}\ \ddot{\boldsymbol{x}} + \boldsymbol{H}(\bar{\boldsymbol{x}}, \dot{\bar{\boldsymbol{x}}}) = \boldsymbol{f}(t) \qquad (1)$$

where

$$\boldsymbol{M}=\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & m_{\mathrm{n+1}} \end{bmatrix}, \quad \boldsymbol{f}(t)=[\,f_1(t) \quad f_2(t) \quad \cdots \quad f_{\mathrm{n+1}}(t)]^{\mathrm{T}}, \tag{2}$$

$$\boldsymbol{H}(\bar{\boldsymbol{x}},\dot{\bar{\boldsymbol{x}}})=[H(\bar{x}_1,\dot{\bar{x}}_1) \quad H(\bar{x}_2,\dot{\bar{x}}_2) \quad \cdots \quad H(\bar{x}_{\mathrm{n}},\dot{\bar{x}}_{\mathrm{n}})]^{\mathrm{T}}.$$

The nonlinear function $H(\bar{x}_i,\dot{\bar{x}}_i)$ may be decomposed into stiffness $H_{\mathrm{S}}(\bar{x}_i)$ and viscous damping $H_{\mathrm{D}}(\dot{\bar{x}}_i)$ characteristics. In particular, the stiffness function is related to the force $F_{\mathrm{S}\,i}=k_i\,h_i(\bar{x}_i)$ where k_i is the stiffness coefficient and $h_i(\bar{x}_i)$ is the displacement nonlinearity in clearance. The viscous damping function is assumed to be linear, so the damping force is given by $c_i\,\dot{\bar{x}}_i$ where c_i is the damping coefficient. Therefore, the equation of motion, Eq. (1), may be rewritten as

$$\boldsymbol{M}\,\ddot{\bar{\boldsymbol{x}}}+\boldsymbol{C}\,\dot{\bar{\boldsymbol{x}}}+\boldsymbol{K}\,\boldsymbol{h}(\bar{\boldsymbol{x}})=\boldsymbol{f}(t) \tag{3}$$

where

$$\boldsymbol{C}=\begin{bmatrix} c_1 & 0 & \vdots & 0 & 0 \\ -c_1 & c_2 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & -c_{\mathrm{n}-1} & c_{\mathrm{n}} \\ 0 & 0 & \vdots & 0 & -c_{\mathrm{n}} \end{bmatrix}, \quad \boldsymbol{K}=\begin{bmatrix} k_1 & 0 & \vdots & 0 & 0 \\ -k_1 & k_2 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & -k_{\mathrm{n}-1} & k_{\mathrm{n}} \\ 0 & 0 & \vdots & 0 & -k_{\mathrm{n}} \end{bmatrix},$$

$$\boldsymbol{h}(\bar{\boldsymbol{x}})=[h_1(\bar{x}_1) \quad h_2(\bar{x}_2) \quad \cdots \quad h_{\mathrm{n}}(\bar{x}_{\mathrm{n}})]^{\mathrm{T}}. \tag{4}$$

By introducing the nondimensional relative displacement $\boldsymbol{q}$ and the nondimensional time $\tau=\omega_k\,t$ as new independent variables, Eq. (3) can be condensed into nondimensional form

$$\boldsymbol{q}''+\boldsymbol{Z}\boldsymbol{q}'+\boldsymbol{\Omega}\,\boldsymbol{h}(\boldsymbol{q})=\boldsymbol{f}_m+\boldsymbol{f}_a\cos(\eta\tau) \tag{5}$$

where

$$\boldsymbol{q}=\frac{1}{b}\bar{\boldsymbol{x}}, \quad \omega_k=\sqrt{k_1\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}(\,)=(\,)',$$

$$\boldsymbol{Z} = 2\begin{bmatrix} \zeta_{11} & -\zeta_{12}\omega_{12} & \vdots & 0 \\ -\zeta_{21}\omega_{21} & \zeta_{22}\omega_{22} & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & \zeta_{nn}\omega_{nn} \end{bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} 1 & -\omega_{12}^2 & \vdots & 0 \\ -\omega_{21}^2 & \omega_{22}^2 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & \omega_{nn}^2 \end{bmatrix}. \tag{6}$$

Furthermore, b is the characteristic length, η denotes a nondimensional excitation frequency while $\boldsymbol{f}_m$ and $\boldsymbol{f}_a$ are the amplitude vectors of mean and alternating load, respectively. The elements of matrices $\boldsymbol{Z}$ and $\boldsymbol{\Omega}$ are found to be

$$\zeta_{11} = \frac{c_1\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}{2\omega_k}, \quad \zeta_{22} = \frac{c_2\left(\frac{1}{m_2}+\frac{1}{m_3}\right)}{2\,\omega_k\,\omega_{22}}, \quad \zeta_{nn} = \frac{c_n\left(\frac{1}{m_n}+\frac{1}{m_{n+1}}\right)}{2\,\omega_k\,\omega_{nn}},$$

$$\zeta_{12} = \frac{c_2}{2\,m_2\,\omega_k\,\omega_{12}}, \quad \zeta_{21} = \frac{c_1}{2\,m_2\,\omega_k\,\omega_{21}}, \tag{7}$$

$$\omega_{12}^2 = \frac{k_2}{m_2\,\omega_k^2}, \quad \omega_{21}^2 = \frac{k_1}{m_2\,\omega_k^2}, \quad \omega_{22}^2 = \frac{k_2\left(\frac{1}{m_2}+\frac{1}{m_3}\right)}{\omega_k^2}, \quad \omega_{nn}^2 = \frac{k_n\left(\frac{1}{m_n}+\frac{1}{m_{n+1}}\right)}{\omega_k^2}.$$

The piecewise linear unit displacement function $h_i(q_i)$ for $i = 1(1)n$ (Fig. 2) which describes the clearance of value 2b is defined as

$$h_i(q_i) = q_i + \frac{\mathrm{abs}(q_i - 1) - \mathrm{abs}(q_i + 1)}{2}. \tag{8}$$

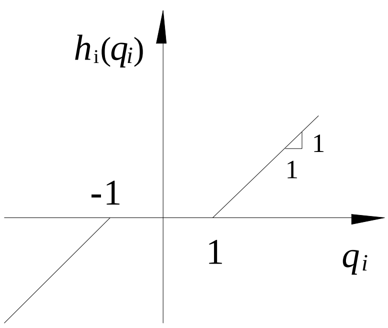

Fig. 2 Piecewise linear unit displacement function

3. THE NOVEL ALGORITHM

The novel algorithm for solving MDOF system with clearances is an extension of the classical method of piecing the exact solutions. The method of piecing the exact solutions is not applicable for MDOF system due to the unexpected complexity of solutions. The proposed algorithm is based on the substitution of nonlinear term in Eq. (5) with a set of linear equations defined within each of the piecewise linear stage stiffness regions. As the clearance is modeled with three linear domains, Fig. 2, the nonlinear term has to be replaced with 3^n linear equations as follows

$$\boldsymbol{\Omega}\,\boldsymbol{h}(\boldsymbol{q}) = \bar{\boldsymbol{\Omega}}_j\,\boldsymbol{q} - \boldsymbol{b}_j\,, \quad j = 1(1)3^{\mathrm{n}} \tag{9}$$

where $\bar{\boldsymbol{\Omega}}_j$ is the local stiffness matrix and $\boldsymbol{b}_j$ denotes the breakpoint vector. Regarding Eq. (9), the equation of motion becomes a set of 3^n linear equations of motion

$$\boldsymbol{q}'' + \boldsymbol{Z}\boldsymbol{q}' + \bar{\boldsymbol{\Omega}}_j\,\boldsymbol{q} = \boldsymbol{b}_j + \boldsymbol{f}_m + \boldsymbol{f}_a\cos(\eta\tau), \quad j = 1(1)3^{\mathrm{n}}\,. \tag{10}$$

The nonlinear system starts from an initial position described with one of the local equations of motion. When the system changes a stage stiffness region, the system is represented with the new local equation of motion. Determination of times in which the system changes a stiffness region can be done only numerically.

Linear equations of motion can be solved by applying many mathematical methods. Using the state-space formulation, preferable in the time-domain analysis, Eq. (10) will be reduced to first-order differential equations of the form

$$\boldsymbol{y}_j' = \boldsymbol{A}_j\,\boldsymbol{y}_j + \boldsymbol{B}\,\boldsymbol{u}_j\,, \quad j = 1(1)3^{\mathrm{n}} \tag{11}$$

where $\boldsymbol{y}_j = \begin{bmatrix} \boldsymbol{q}_j \\ \boldsymbol{q}_j' \end{bmatrix}$ is the state vector, $\boldsymbol{A}_j = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\bar{\boldsymbol{\Omega}}_j & -\boldsymbol{Z} \end{bmatrix}$ is the state matrix, $\boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}$ is the input gain matrix while $\boldsymbol{u}_j = \boldsymbol{b}_j + \boldsymbol{f}_m + \boldsymbol{f}_a\cos(\eta\tau)$ denotes the excitation vector. $\boldsymbol{0}$ and $\boldsymbol{I}$ are referred to null and identity matrices, respectively. A similarity transformation

$$\boldsymbol{V}^{-1}\boldsymbol{A}\boldsymbol{V} = \boldsymbol{\Lambda} \tag{12}$$

with the matrices of eigenvalues $\boldsymbol{\Lambda} = \mathbf{diag}(\lambda_k), \ \ k = 1(1)2\mathrm{n}$ and the eigenvectors $\boldsymbol{V}$, allows a coordinate transformation

$$\boldsymbol{y}_j = \boldsymbol{V}_j\,\boldsymbol{z}_j \tag{13}$$

which uncouples Eq. (11) giving

$$\boldsymbol{z}_j' = \boldsymbol{\Lambda}_j\,\boldsymbol{z}_j + \boldsymbol{g}_j\,. \tag{14}$$

In Eq. (14), $\boldsymbol{z}_j$ is the normal coordinate and $\boldsymbol{g}_j = \boldsymbol{V}_j^{-1}\boldsymbol{B}\boldsymbol{u}_j$. Each row of Eq. (14) has the form

$$z_k' = \lambda_k\, z_k + g_k\,, \quad k = 1(1)2\mathrm{n} \tag{15}$$

with

$$g_k = \bar{f}_{\mathrm{m}k} + \bar{f}_{\mathrm{a}k}\cos(\eta\tau + \bar{\varphi}_k)\,. \tag{16}$$

Eq. (15) has a well known analytical solution. The accuracy of the method does not depend on a magnitude of the integration step since the local solutions are obtained analytically. The time step has to be sufficiently small for a reliable numerical determination of the points of entering in each stage stiffness region. It is a remarkable advantage with respect to other numerical integration methods.

In explicit integration methods, the stability of solutions cannot be studied considering the standard stability procedures such as Poincaré map or Floquet theory. The responses obtained by the proposed algorithm can be only classified as either periodic or nonperiodic. Nonperiodic responses may correspond to a quasiperiodic, transient or chaotic motion. If the alternating amplitude calculated by the relation

$$q_a = \frac{q_{max} - q_{min}}{2} \tag{17}$$

coincides with the effective amplitude (root-mean-square value)

$$q_{\mathrm{ef}} = \sqrt{\frac{1}{\pi}\int_0^{2\pi}(q(\tau) - q_{\mathrm{av}})^2\mathrm{d}\tau}\,, \tag{18}$$

the response is periodic; otherwise the response is nonperiodic. In Eq. (18), q_{av} is the average of the $q(\tau)$

$$q_{\mathrm{av}} = \frac{1}{2\pi}\int_0^{2\pi} q(\tau)\,\mathrm{d}\tau\,. \tag{19}$$

3. NUMERICAL EXAMPLE

The three-degree-of-freedom semi-definite system with two clearances, shown in Fig. 3, is studied to demonstrate the application of novel algorithm. The system parameters $\zeta_{11}=\zeta_{12}=\zeta_{21}=\zeta_{22}=0.05$, $\omega_{12}=\omega_{21}=0.6$, $\omega_{22}=1.1$, $\boldsymbol{f}_{\mathrm{m}}^{\mathrm{T}}=[0.5,0.25]$ and $\boldsymbol{f}_{\mathrm{a}}^{\mathrm{T}}=[0.25,0]$ are adopted from (Padmanabhan 1992) where the system with one clearance was studied.

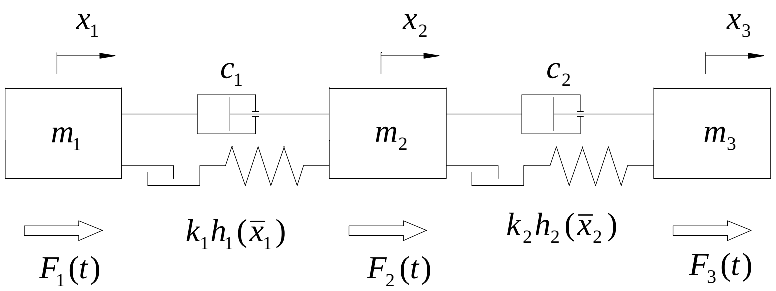

Fig. 3 Three-degree-of-freedom semi-definite system with two clearances

According to the proposed algorithm, the procedure requires solving the following set of equations of motion

$$\boldsymbol{q}''+\boldsymbol{Z}\boldsymbol{q}'+\bar{\boldsymbol{\Omega}}_j\,\boldsymbol{q}=\boldsymbol{b}_j+\boldsymbol{f}_m+\boldsymbol{f}_a\cos(\eta\tau),\quad j=1(1)9\,. \tag{20}$$

The local stiffness matrices $\bar{\boldsymbol{\Omega}}_j$ and the vectors of breakpoint take the forms

$$\begin{aligned}
&\bar{\boldsymbol{\Omega}}_1=\bar{\boldsymbol{\Omega}}_3=\bar{\boldsymbol{\Omega}}_7=\bar{\boldsymbol{\Omega}}_9=\begin{bmatrix}1 & -\omega_{12}^2\\ -\omega_{21}^2 & \omega_{22}^2\end{bmatrix},\\
&\bar{\boldsymbol{\Omega}}_2=\bar{\boldsymbol{\Omega}}_8=\begin{bmatrix}1 & 0\\ -\omega_{21}^2 & 0\end{bmatrix},\\
&\bar{\boldsymbol{\Omega}}_4=\bar{\boldsymbol{\Omega}}_6=\begin{bmatrix}0 & -\omega_{12}^2\\ 0 & \omega_{22}^2\end{bmatrix},\quad \bar{\boldsymbol{\Omega}}_5=\boldsymbol{0}\\
&b_1^{\mathrm{T}}=-b_9^{\mathrm{T}}=\left[(1-\omega_{12}^2),\quad (-\omega_{21}^2+\omega_{22}^2)\right],\\
&b_2^{\mathrm{T}}=-b_8^{\mathrm{T}}=\left[1,-\omega_{21}^2\right],\\
&b_3^{\mathrm{T}}=-b_7^{\mathrm{T}}=\left[(1+\omega_{12}^2),\quad -(\omega_{21}^2+\omega_{22}^2)\right],\\
&b_4^{\mathrm{T}}=-b_6^{\mathrm{T}}=\left[-\omega_{12}^2,\omega_{22}^2\right],\quad b_5^{\mathrm{T}}=[0,0].
\end{aligned} \tag{21}$$

Starting from the trivial initial condition, the computations are performed simulating 128 excitation periods for the single point of excitation frequency. This procedure is well suited (Thompson 1986) because no another reliable way to distinguish the transient and steady state motion at chaotic responses. The switch points in each stage stiffness region and the eigenvalues of the state matrix were computed employing the MATLAB routines FZERO and EIG. The frequency responses of the first and second displacements are presented in Figs. 4 and 5. The nonperiodic solutions are found in the frequency range 0.77 < η < 0.94; in other frequency spans the alternating and effective amplitudes are the same, which means that the frequency responses are periodic.

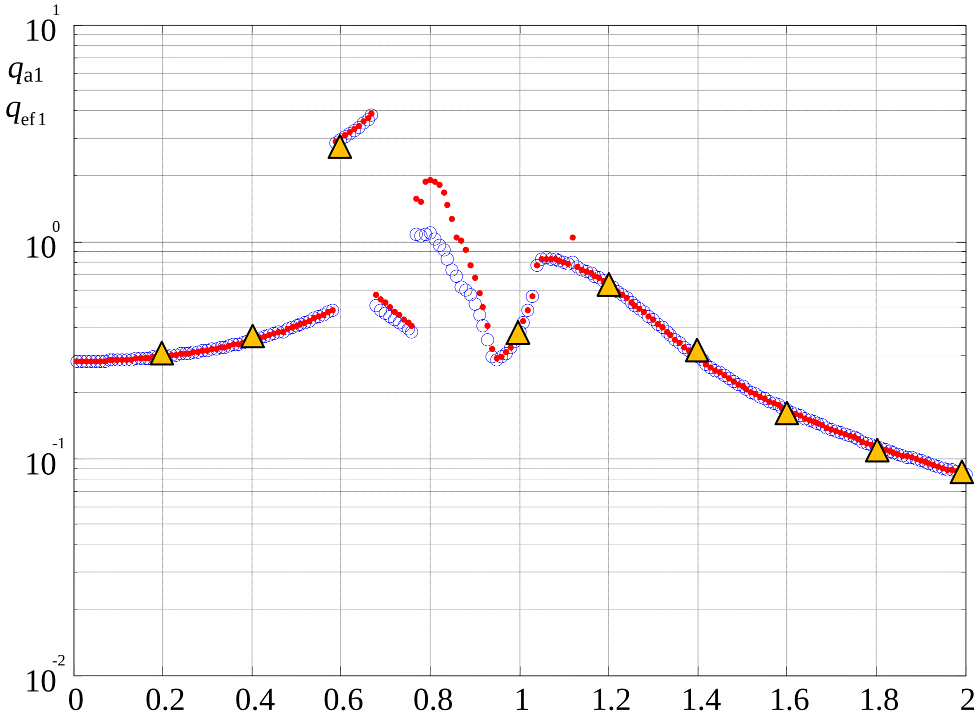

Fig. 4 Frequency response of the first displacement;
● alternating amplitude q_{a1}, ○ effective amplitude q_{ef1} (novel algorithm),
△ alternating amplitude q_{a1} (ODE113)

Responses of dynamical system with clearances can be also analyzed considering MATLAB Simulink software package which simulates a dynamical system by computing its states at successive time steps over a specified time span, using information provided by the model. Simulink provides a wide variety of numerical integration techniques. Numerical integration solutions will be considered to be strictly valid only in regions with just one steady state solution. When two steady state solutions exist, the numerical integration finds the solution with the smaller amplitude of vibration.

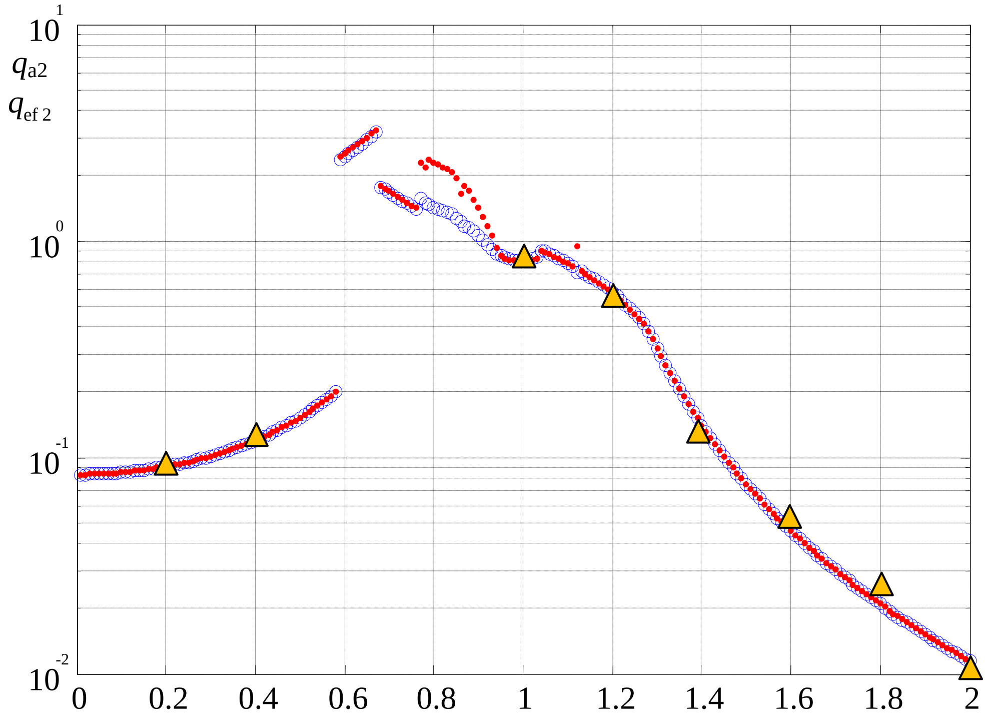

Fig. 5 Frequency response of the second displacement;
● alternating amplitude q_{a2}, ○ effective amplitude $q_{ef\,2}$ (novel algorithm),
△ alternating amplitude q_{a2} (ODE113)

To evaluate the results obtained by the novel algorithm, the nonlinear equation of motion, Eq. (5) was solved by using Simulink ODE113 solver. ODE113 is a multistep solver based on Adams-Bashforth-Moulton predict-evaluate-correct-evaluate mode. The obtained amplitudes agree very well with the results obtained by the proposed algorithm.

4. CONCLUSIONS

The novel algorithm is a robust numerical procedure for predicting the steady state response of dynamical systems with clearances. The algorithm is an extension of the method of piecing the exact solutions. The accuracy of displacements and velocities, at any point of each linear stage, is the same because the algorithm is based on semi-analytical solutions. It is a significant advantage with respect to other explicit integration methods (Runge-Kutta, etc).

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