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Nonlocal plasticity and damage

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ABSTRACT

A thermodynamically consistent nonlocal theory of elastoplasticity coupled with damage in the strain space is presented. The theory is developed in the framework of the generalized standard material and the constitutive model is identified by the specification of the internal energy and the use of the first law of thermodynamics in a nonlocal form and of the classical second law of thermodynamics. The nonlocal constitutive model is then addressed and a variational formulation depending on the complete set of local and nonlocal state variables is provided.

1. INTRODUCTION

The effective stress concept was introduced by Kachanov (1958) to provide a phenomenological damage model for the isotropic case.

In the early seventies a model of generalized standard elastoplastic material has been proposed by Halphen and Nguyen (1975). In that model the flow rule is assigned by a normality rule to a generalized elastic domain defined in the product space of stresses and thermodynamic forces.

Based on these proposals, several models of plasticity with internal variables coupled with damage have been presented in literature in the framework of continuum mechanics.

The necessity for introducing the nonlocal integral or gradient theory in a continuum model stems from the well-known circumstance that the classical rate-independent plasticity and damage theories do not possess an intrinsic length scale.

This fact leads to numerical stability problems such as mesh size and mesh sensitivity in problems exhibiting strain localization phenomena or scale effects. However, several regularization approaches have been proposed in the constitutive modeling to overcome these shortcomings such as viscoplastic models (see e.g. Perzyna, 1963; Glema et al., 2000), nonlocal models (e.g. Pijaudier-Cabot and Bazant, 1987; Voyiadjis and Dorgan, 2001; Marotti de Sciarra, 2009a, Thai, 2012) and strain-gradient models (e.g. de Borst and Muhlhaus, 1992; Wang et al., 2003).

In the present paper a phenomenological model for a class of nonlocal elastoplastic damaging materials is addressed considering a nonlocal model of integral type.

The aim of the paper is to formulate a nonlocal elastoplastic model, defined in the generalized stress space, coupled with nonlocal damage defined in the strain space. The nonlocal elastoplastic formulation as well as the stress decomposition of the nonlocal strain damage behaviour consistently follows from the thermodynamic analysis in a nonlocal integral context.

The second objective of this paper is to stem a general thermodynamic framework which provides the tools to derive a consistent variational formulation for the nonlocal constitutive problem of elastoplasticity coupled with damage in the strain space.

The nonlocal constitutive equations are derived from a nonlocal form of the first law of thermodynamics, from the classical form of the second law and a suitable expression of the internal energy. Accordingly the nonlocal counterpart of the Clausius–Duhem inequality is obtained and the maximum dissipation principle for the nonlocal elastoplastic damage problem is straightforward obtained.

Finally the variational formulation associated with the nonlocal constitutive model is provided for the first time.

From a computational point of view, an advantage of models with strain-based damage is that the stress corresponding to a given strain can be evaluated directly without any need for solving a nonlinear system of equations (see e.g. Marotti de Sciarra, 2009b).

2. NONLOCAL CONSTITUTIVE FRAMEWORK

A quasi-static evolution process in a geometrically linear range is considered for a body defined on a regular bounded domain Ω of an Euclidean space. The inelastic model is subjected to a given load history and the time is conceived as a monotonically increasing parameter which orders successive events so that a time-independent mechanical behaviour of the body is assumed.

The classical theory of small deformation plasticity is based on the additive decomposition of the total strain $\boldsymbol{\varepsilon}$ into elastic $\mathbf{e}$ and plastic $\mathbf{p}$ parts, with $\mathbf{e}$ being the elastic component and $\mathbf{p}$ being the corresponding plastic strain.

A model governing a nonlocal stress-based plasticity with a nonlocal strain-based damage is developed following the generalized standard material and the constitutive framework presented in Marotti de Sciarra (2009b) in the case of local plasticity. The plastic and damage state of the body is phenomenologically described by a set of internal variables and by the related mechanisms for energy exchange. Reversible phenomena modify the stored energy and the irreversible phenomena induce energy dissipation.

The evolution of the hardening phenomena of associated type is described in terms of a set of dual kinematic and static internal variables which account for the changes in the material structure at the microscale level (Halphen and Nguyen, 1975). The dual set of internal variables are reported in the following Table 1.

The back-stresses $\boldsymbol{\chi}_1$ and $\boldsymbol{\chi}_2$ are associated with the kinematic hardening, the drag-stresses χ_3, χ_4 and χ_5 are associated with isotropic hardening/softening.

Classical inelastic theories are unable to describe the softening behaviour, small-scale phenomena or the effect of relative size on the mechanical properties of the material since such theories do not possess an intrinsic material length scale. These problems can then be avoided with the use of a nonlocal theory which introduces the length scale in the constitutive equations.

Table 1 The dual set of kinematic and static variables

State variables		Associated conjugates
Observable	Internal	
$\boldsymbol{\varepsilon}$		$\boldsymbol{\sigma}$, $\mathbf{s}$, $\boldsymbol{\sigma}^e$
	$\mathbf{p}$	$\boldsymbol{\chi}$
	$\boldsymbol{\alpha}_1$	$\boldsymbol{\chi}_1$
	$\boldsymbol{\alpha}_2$	$\boldsymbol{\chi}_2$
	$\boldsymbol{\alpha}_3$	$\boldsymbol{\chi}_3$
	$\boldsymbol{\alpha}_4$	$\boldsymbol{\chi}_4$
	$\boldsymbol{\alpha}_5$	$\boldsymbol{\chi}_5$

The nonlocal behaviour associated with plasticity is governed by the nonlocal field $R\boldsymbol{\alpha}_2$ which can be obtained as a spatial weighted average of the local variable $\boldsymbol{\alpha}_2$ by the following parametric relation:

$$\bar{\boldsymbol{\alpha}}_2(x) = R\boldsymbol{\alpha}_2(x) = \int_\Omega W(x,y)\boldsymbol{\alpha}_2(y)\,dy \quad (1)$$

where $W(x,y)$ is the weight function.

Similar relations hold for the nonlocal counterparts of the kinematic internal variables $\boldsymbol{\alpha}_3$ and $\boldsymbol{\alpha}_5$:

$$\bar{\boldsymbol{\alpha}}_3(x) = R\boldsymbol{\alpha}_3(x) = \int_\Omega W(x,y)\boldsymbol{\alpha}_3(y)\,dy,\ \bar{\boldsymbol{\alpha}}_5(x) = R\boldsymbol{\alpha}_5(x) = \int_\Omega W(x,y)\boldsymbol{\alpha}_5(y)\,dy. \quad (2)$$

The long range forces arising in a damaged structure are provided by the nonlocal static internal variable $\bar{\mathbf{s}}$ which has the mechanical meaning of the nonlocal relaxation stress as shown in the sequel. It is expressed in the following form:

$$\bar{\mathbf{s}}(x) = R\mathbf{s}(x) = \int_\Omega W(x,y)\mathbf{s}(y)\,dy. \quad (3)$$

In this paper is not necessary to give an explicit expression to the weight functions which will be left unspecified.

Generally the deformation process in metals enhances the creation, motion, and storage of the dislocations. In particular, the material hardening is caused by the storage of dislocations which can be referred to as statistically-stored dislocations (SSDs) and geometrically-necessary dislocations (GNDs). The SSDs are generated by trapping the dislocations each other in a random way and the GNDs represent the stored dislocations which are required for compatible deformations within the polycrystals. During plastic deformations, the density of SSDs increases due to a wide range of processes that lead to the production of new dislocations. Those new generated dislocations travel on a background of GNDs which causes additional storage of defects and increases the deformation resistance by acting like obstacles to the SSDs (Gao et al., 1999).

The SSDs and GNDs are different in nature since experimental evidences show that the SSDs dependent on the effective plastic strain while the GNDs are associated with the gradient of the effective plastic strain (Ashby, 1970; Fleck and Hutchinson, 1997; Arsenlis and Parks, 1999; Gao et al. 1999).

Moreover it is shown in, e.g., Abu Al-Rub and Voyiadjis (2003), Fleck and Hutchinson (2001) that the multiplicity of plastic phenomena at small-scale levels suggests the use of more than one length parameter in the nonlocal gradient description.

However, in this paper the two inelastic behaviors are modeled by nonlocal relations of integral type depending on a unique internal length scale. Nevertheless the model can be enhanced, in a forthcoming paper, by considering that the nonlocalities due to plasticity and damage be governed by different length scales.

The determination of the evolution of the assumed internal state variables is one of the main challenge of the constitutive modeling. This task can be effectively achieved through the thermodynamic principles for the development of a continuum thermo-elasto-plastic-damage based model where damage is modeled in the strain space.

The first principle of thermodynamics (Lemaitre and Chaboche, 1994) for a nonlocal model and isothermal processes is expressed pointwise in the following form, see e.g. Edelen and Laws (1971):

$$\dot{u} = \boldsymbol{\sigma} * \boldsymbol{\varepsilon} + \dot{Q} + P \tag{4}$$

where the explicit dependence on the point has been dropped for simplicity. The heat supplied to an element of volume is $\dot{Q}$=-div**q**, being **q** the heat flux, and $\boldsymbol{\sigma}$ is the actual stress. The internal energy density *u* depends on elastic and plastic strain, on the kinematic internal variables $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_4$ and on the kinematic nonlocal internal variables $R\boldsymbol{\alpha}_2, R\boldsymbol{\alpha}_3, R\boldsymbol{\alpha}_5$.

The nonlocality residual function *P* takes into account the energy exchanges between neighbor particles, see e.g. Edelen and Laws (1971).

Being the body a thermodynamically isolated system with reference to energy exchanges due to nonlocality, the following insulation condition holds:

$$\int_{\Omega} P(x)\, dx = 0. \tag{5}$$

The second principle of thermodynamics for a nonlocal behaviour, is written in its classical pointwise form, see Marotti de Sciarra (2009a) for a discussion:

$$\dot{s}T + div\boldsymbol{q} - \nabla T * \frac{\boldsymbol{q}}{T} \geq 0 \tag{6}$$

in any point of the body where $\dot{s}$ is the internal entropy production rate per unit volume.

Considering isothermal processes, the total dissipation is given by:

$$D = \boldsymbol{\sigma} * \boldsymbol{\varepsilon} - \dot{\phi} + P \geq 0 \tag{7}$$

where ϕ denotes the free energy.

The complexity of the constitutive model is directly determined by the form of the free energy so that its definition constitutes a crucial point of the formulation.

Accordingly the derivation of the constitutive equations is based on the assumed expression of the free energy:

$$\phi(\boldsymbol{\varepsilon}, \mathbf{p}, \boldsymbol{\alpha}_1, R\boldsymbol{\alpha}_2, R\boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, R\boldsymbol{\alpha}_5) = \psi(\boldsymbol{\varepsilon} - \mathbf{p}, \boldsymbol{\alpha}_1, R\boldsymbol{\alpha}_2, R\boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, R\boldsymbol{\alpha}_5) - R\mathbf{s} * \boldsymbol{\varepsilon} \tag{8}$$

where the elastic energy ψ depends on the difference between the total strain and the plastic strain and on the kinematic local and nonlocal internal variables. Moreover, the additive term in the free energy ϕ is based on the fact that damage has a distinctive morphology which is different from the plastic deformation mechanisms. Furthermore, since plasticity and damage correspond to different mechanisms, it will be shown in the sequel that the expression (8) allows us to derive a yield criterion in the stress space and a damage domain in the strain space and two evolutive relations can then be defined.

Expanding the inequality (7) and substituting the rate of the expression (8) above of the free energy, one obtains the following thermodynamic constraints:

$$D = (\boldsymbol{\sigma} + \bar{\mathbf{s}} - d_{\boldsymbol{\varepsilon}}\phi) * \dot{\boldsymbol{\varepsilon}} - d_{\mathbf{p}}\phi * \dot{\mathbf{p}} - d_{\boldsymbol{\alpha}_1}\phi * \dot{\boldsymbol{\alpha}}_1 - d_{\bar{\boldsymbol{\alpha}}_2}\phi * \dot{\bar{\boldsymbol{\alpha}}}_2 - d_{\bar{\boldsymbol{\alpha}}_3}\phi * \dot{\bar{\boldsymbol{\alpha}}}_3 - d_{\boldsymbol{\alpha}_4}\phi * \dot{\boldsymbol{\alpha}}_4 + -d_{\bar{\boldsymbol{\alpha}}_5}\phi * \dot{\bar{\boldsymbol{\alpha}}}_5 + P \geq 0 \qquad (9)$$

Assuming that the axiom of entropy production holds, then the above inequality results in the thermodynamic state laws reported in the following Table 2:

Table 2 The thermodynamic state laws

		$\boldsymbol{\sigma}^e = d_{\boldsymbol{\varepsilon}}\phi$
		$\boldsymbol{\chi} = d_{\mathbf{p}}\phi = -\boldsymbol{\sigma}^e$
Plasticity	Kinematic hardening	$\boldsymbol{\chi}_1 = d_{\boldsymbol{\alpha}_1}\phi$
		$\boldsymbol{\chi}_2 = d_{\bar{\boldsymbol{\alpha}}_2}\phi$
	Isotropic hardening	$\boldsymbol{\chi}_3 = d_{\bar{\boldsymbol{\alpha}}_3}\phi$
		$\boldsymbol{\chi}_4 = d_{\boldsymbol{\alpha}_4}\phi$
		$-\boldsymbol{\chi}_5 = d_{\bar{\boldsymbol{\alpha}}_5}\phi$
Damage		$\boldsymbol{\sigma} = \boldsymbol{\sigma}^e - \bar{\mathbf{s}}$

The relations in Table 2 describe the links between the state variables and their associated thermodynamic conjugate forces. The stress $\boldsymbol{\sigma}$ is a measure of the elastic changes in the internal structure, the elastic stress $\boldsymbol{\sigma}^e$ is related to the elastic strain and represents the stress in an undamaged body, the variables $\boldsymbol{\chi}_i$ (i = 1,…,5) are the conjugate forces corresponding to the plastic internal state variables $(R_i\boldsymbol{\alpha}_i, R\boldsymbol{\alpha}_5)$ with i = 1,…,4, where $\{R_i\} = \{I, R, R, I\}$. The conjugate forces $\boldsymbol{\chi}_i$ (i = 1,…,5) are measures of plastic changes in the internal structure and the relaxation stress $\mathbf{s}$ is a measure of the damage changes in the internal structure.

The sum of the actual stress $\boldsymbol{\sigma}$ and of the nonlocal stress $R\mathbf{s}$ provides the elastic stress $\boldsymbol{\sigma}^e$. From a mechanical point of view the nonlocal stress $R\mathbf{s}$ yields the total damage which is accumulated during loading and the elastic stress causes the same state of deformation in a virgin material as in a damaged material. The relationships between the proposed damage model and the classical one are analyzed in Marotti de Sciarra (2009b). The nonlocal stress $R\mathbf{s}$ is then associated with the elastic-damage changes in the internal structure resulting from crack and voids during the loading process.

At every point where an irreversible mechanism develops, the dissipation can be assumed in a bilinear form and following the procedure reported in Marotti de Sciarra (2008), the nonlocal counterpart of the Clausius–Duhem inequality expresses the fact that the dissipation energy *D* is necessarily nonnegative as follows:

$$D = \boldsymbol{\sigma}^e * \dot{\mathbf{p}} - \boldsymbol{\chi}_1 * \dot{\boldsymbol{\alpha}}_1 - R\boldsymbol{\chi}_2 * \dot{\boldsymbol{\alpha}}_2 - R\boldsymbol{\chi}_3 * \dot{\boldsymbol{\alpha}}_3 - \boldsymbol{\chi}_4 * \dot{\boldsymbol{\alpha}}_4 + R\boldsymbol{\chi}_5 * \dot{\boldsymbol{\alpha}}_5 + \dot{\mathbf{s}} * R\boldsymbol{\varepsilon} \geq 0. \qquad (10)$$

This result requires that the inelastic work is dissipated away as heat, except for that energy which is stored because of the rearrangement of the material internal structure. Although the dissipation D is written in the decoupled form as shown by Eq. (10), the corresponding physical mechanisms is not decoupled. In fact coupling does occur between plasticity and damage since the conjugate forces and their associated fluxes are related each other so that two damage mechanisms are introduced: one mechanism is coupled with plasticity and the other can occur independent of plastic deformation.

Finally the nonlocality residual function has the following expression:

$$P = \boldsymbol{\chi}_2 * R\dot{\boldsymbol{\alpha}}_2 - R\boldsymbol{\chi}_2 * \dot{\boldsymbol{\alpha}}_2 + \boldsymbol{\chi}_3 * R\dot{\boldsymbol{\alpha}}_3 - R\boldsymbol{\chi}_3 * \dot{\boldsymbol{\alpha}}_3 - \boldsymbol{\chi}_5 * R\dot{\boldsymbol{\alpha}}_5 + \\ + R\boldsymbol{\chi}_5 * \dot{\boldsymbol{\alpha}}_5 - R\dot{\mathbf{s}} * \boldsymbol{\varepsilon} + \dot{\mathbf{s}} * R\boldsymbol{\varepsilon}. \quad (11)$$

Hence the presented approach has introduced a way for incorporating nonlocality, based on the concept of nonlocality residual, into an existing thermodynamic framework.

3. PLASTIC AND DAMAGE EVOLUTION CRITERIA

The plastic and damage evolution criteria for the considered nonlocal model can be obtained by assuming that the sublinear function D is lower-semicontinuous (Rockafellar, 1970).

Hence the dissipation D turns out to be the support function of a closed convex domain C which is given by the local and nonlocal state variables such that the following inequality is fulfilled:

$$D \geq \boldsymbol{\sigma}^{\boldsymbol{e}} * \dot{\mathbf{p}} - \boldsymbol{\chi}_1 * \dot{\boldsymbol{\alpha}}_1 - R\boldsymbol{\chi}_2 * \dot{\boldsymbol{\alpha}}_2 - R\boldsymbol{\chi}_3 * \dot{\boldsymbol{\alpha}}_3 - \boldsymbol{\chi}_4 * \dot{\boldsymbol{\alpha}}_4 + R\boldsymbol{\chi}_5 * \dot{\boldsymbol{\alpha}}_5 + \dot{\mathbf{s}} * R\boldsymbol{\varepsilon} \quad (12)$$

for any $(\dot{\mathbf{p}}, -\dot{\boldsymbol{\alpha}}_1, -\dot{\boldsymbol{\alpha}}_2, -\dot{\boldsymbol{\alpha}}_3, -\dot{\boldsymbol{\alpha}}_4, \dot{\boldsymbol{\alpha}}_5, \dot{\mathbf{s}})$.

In mechanical terms, the domain C is the set of admissible elastic stresses, local and nonlocal static internal variables and nonlocal strains and its boundary represents the elasto-damage surface.

As a consequence such a model provides a unique generalized evolutive relation for plasticity and damage.

Since plasticity and damage correspond to different mechanisms acting on different scales, the two inelastic models have to be modeled by different evolution equations. Accordingly the proposed nonlocal elastoplastic-damage model is now specialized to a nonlocal elastoplastic model and to a nonlocal damage model depending on two evolutive relations.

To this end the dissipation D can be splitted into two parts which are associated with the plastic and the damage mechanisms:

$$\begin{cases} D_1 = \boldsymbol{\sigma}^{\boldsymbol{e}} * \dot{\mathbf{p}} - \boldsymbol{\chi}_1 * \dot{\boldsymbol{\alpha}}_1 - R\boldsymbol{\chi}_2 * \dot{\boldsymbol{\alpha}}_2 - R\boldsymbol{\chi}_3 * \dot{\boldsymbol{\alpha}}_3 - \boldsymbol{\chi}_4 * \dot{\boldsymbol{\alpha}}_4 + R\boldsymbol{\chi}_5 * \dot{\boldsymbol{\alpha}}_5 \geq 0 \\ D_2 = \dot{\mathbf{s}} * R\boldsymbol{\varepsilon} \geq 0. \end{cases} \quad (13)$$

The dissipations $D_1(\dot{\mathbf{p}}, -\dot{\boldsymbol{\alpha}}_1, -\dot{\boldsymbol{\alpha}}_2, -\dot{\boldsymbol{\alpha}}_3, -\dot{\boldsymbol{\alpha}}_4, \dot{\boldsymbol{\alpha}}_5)$ and $D_2(\dot{\mathbf{s}})$ are the support functions of two closed convex domains C_1 and C_2, respectively. From the physical standpoint, C_1 is the set of admissible elastic stresses $\boldsymbol{\sigma}^{\boldsymbol{e}}$, local and nonlocal static internal variables $(R_i\boldsymbol{\chi}_i, R\boldsymbol{\chi}_5)$ (i = 1,…,4) and its boundary represents the elastic surface. The domain C_2 is the set of admissible nonlocal strains $S\boldsymbol{\varepsilon}$ and its boundary represents the damage surface.

It can be proved that the dissipation processes (13) imply the existence of the complementary laws:

$$\begin{cases} \boldsymbol{\sigma}^e \in \partial_{\dot{\mathbf{p}}} D_1, \quad R_i \boldsymbol{\chi}_i \in \partial_{-\dot{\boldsymbol{\alpha}}_i} D_1 \ \ (i = 1, \dots, 4), \ R\boldsymbol{\chi}_5 \in \partial_{\dot{\boldsymbol{\alpha}}_5} D_1 \\ R\boldsymbol{\varepsilon} \in \partial_{\dot{\mathbf{s}}} D_2 \end{cases} \tag{14}$$

being ∂ the subdifferential operator and $\{R_i\} = \{I, R, R, I\}$. The relations (14) are equivalent to the normality laws:

$$\begin{cases} (\dot{\mathbf{p}}, -\dot{\boldsymbol{\alpha}}_i, \dot{\boldsymbol{\alpha}}_5) \in \partial I_{C_1}(\boldsymbol{\sigma}^e, R_i \boldsymbol{\chi}_i, R\boldsymbol{\chi}_5) \ \ with \ (i = 1, \dots, 4), \\ \dot{\mathbf{s}} \in \partial I_{C_2}(R\boldsymbol{\varepsilon}) \end{cases} \tag{15}$$

where the indicator functions of the domains C_1 and C_2, denoted by I_{C_1} and I_{C_2}, are the Fenchel's conjugates of the dissipation functions D_1 and D_2 respectively.

It can be proved that the subdifferentials of the indicator functions I_{C_1} and I_{C_2} coincide to the normal cones to the elastoplastic domain C_1 and to the damage domain C_2, respectively. Accordingly the evolution laws of the flux variables can be expressed in terms of the dual local and nonlocal variables (generalized normality relations) in the following forms:

$$\begin{cases} (\dot{\mathbf{p}}, -\dot{\boldsymbol{\alpha}}_i, \dot{\boldsymbol{\alpha}}_5) \in N_{C_1}(\boldsymbol{\sigma}^e, R_i \boldsymbol{\chi}_i, R\boldsymbol{\chi}_5) \ \ with \ (i = 1, \dots, 4), \\ \dot{\mathbf{s}} \in N_{C_2}(R\boldsymbol{\varepsilon}). \end{cases} \tag{16}$$

Properties of Fenchel's conjugates allow us to state that the nonlocal counterpart of the classical maximum dissipation principle holds for this nonlocal elastoplastic-damage model. The nonlocal maximum dissipation principle states that the actual state of the thermodynamic forces $(\boldsymbol{\sigma}^e, R_i \boldsymbol{\chi}_i, R\boldsymbol{\chi}_5)$, i=1,…,4, is that which maximizes the inelastic dissipation functions over all other possible admissible states.

3.1 Plastic and damage flow rules for the nonlocal model

The associative evolution laws can be obtained in terms of the plastic and damage multipliers by assuming that the admissible elastoplastic and damage domains are defined in terms of the plastic and damage modes:

$$\begin{cases} h(\boldsymbol{\sigma}^e, R_i \boldsymbol{\chi}_i, R\boldsymbol{\chi}_5) = h_1(\boldsymbol{\sigma}^e, \boldsymbol{\chi}_1, R\boldsymbol{\chi}_2) - h_2(R\boldsymbol{\chi}_3, \boldsymbol{\chi}_4, R\boldsymbol{\chi}_5) - \boldsymbol{\sigma}_y \\ g(R\boldsymbol{\varepsilon}) = g_1(R\boldsymbol{\varepsilon}) - \boldsymbol{\varepsilon}_d \end{cases} \tag{17}$$

with i=1,…,4, where $\boldsymbol{\sigma}_y$ is the initial size of the yield surface and $\boldsymbol{\varepsilon}_d$ is the initial damage threshold.

Substituting Eq. (17) in (16), the flow rules (16) can be equivalently rewritten in terms of the plastic and damage multipliers (see Marotti de Sciarra, 2004) to get:

$$\begin{cases} (\dot{\mathbf{p}}, -\dot{\boldsymbol{\alpha}}_i, \dot{\boldsymbol{\alpha}}_5) = \lambda_p dh(\boldsymbol{\sigma}^e, R_i \boldsymbol{\chi}_i, R\boldsymbol{\chi}_5) \\ \dot{\mathbf{s}} = \lambda_d dg(R\boldsymbol{\varepsilon}), \end{cases} \tag{18}$$

with i = 1,…,4, under the complementarity conditions:

$$\begin{cases} \lambda_p \geq 0, h(\boldsymbol{\sigma}^e, R_i \boldsymbol{\chi}_i, R\boldsymbol{\chi}_5) \leq 0, \lambda_p h(\boldsymbol{\sigma}^e, R_i \boldsymbol{\chi}_i, R\boldsymbol{\chi}_5) = 0 \\ \lambda_d \geq 0, g(R\boldsymbol{\varepsilon}) \leq 0, \lambda_d g(R\boldsymbol{\varepsilon}) = 0. \end{cases} \tag{19}$$

Here λ_p is the plastic multiplier and λ_d denotes the damage multiplier.

4. FINITE-STEP NONLOCAL FORMULATION

Let $t_0, t_1, \dots, t_n, t_{n+1} = t_n + \Delta t$ be convenient time instances along the time interval over which the response of the body is sought. Consider the time step $\Delta t = t_{n+1} - t_n$. At the time $t = t_n$ all quantities are known since they are the converged values of the previous step and the solution must be computed at $t = t_{n+1}$ for a given strain increment $\Delta\boldsymbol{\varepsilon}$.

Hence the time discretization is performed according to the Euler backward scheme and the constitutive behavior of the body in the small strain range is governed at the time step *n+1* by the relations reported in Table 2 and in Eqs. (14). The finite-step counterparts of the flow rules for the nonlocal model are enforced at the end of the step according to the relations:

$$\begin{cases} \boldsymbol{\sigma}^e \in \partial_{\mathbf{p}} D_1, \quad R_i \boldsymbol{\chi}_i \in \partial_{-\boldsymbol{\alpha}_i} D_1 \ \ (i = 1, \dots, 4), \ R\boldsymbol{\chi}_5 \in \partial_{\boldsymbol{\alpha}_5} D_1 \\ R\boldsymbol{\varepsilon} \in \partial_{\mathbf{s}} D_2 \end{cases} \tag{20}$$

since the time increment Δt can be dropped being the dissipation a sublinear function.

Therefore the nonlocal potential associated with the nonlocal model can then be directly evaluated and the following variational formulation holds.

- *Nonlocal variational formulation*. The set $(\boldsymbol{\varepsilon}, \mathbf{p}, R_i\boldsymbol{\alpha}_i, R\boldsymbol{\alpha}_5, \mathbf{s})$ is a solution of the saddle problem:

$$stat_{\boldsymbol{\alpha}_5} min_{\boldsymbol{\varepsilon},\mathbf{p},\boldsymbol{\alpha}_i,\mathbf{s}} \Pi(\boldsymbol{\varepsilon}, \mathbf{p}, R_i\boldsymbol{\alpha}_i, R\boldsymbol{\alpha}_5, \mathbf{s}) \tag{21}$$

where

$$\Pi(\boldsymbol{\varepsilon}, \mathbf{p}, \boldsymbol{\alpha}_1, R\boldsymbol{\alpha}_2, R\boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, R\boldsymbol{\alpha}_5, \mathbf{s}) = \int_\Omega \psi(\boldsymbol{\varepsilon} - \mathbf{p}, \boldsymbol{\alpha}_1, R\boldsymbol{\alpha}_2, R\boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, R\boldsymbol{\alpha}_5)\, dx +$$

$$+ \int_\Omega D_1(\Delta\mathbf{p}, -\Delta\boldsymbol{\alpha}_i, \Delta\boldsymbol{\alpha}_5)\, dx + \int_\Omega D_2(\Delta\mathbf{s}) dx - \int_\Omega \mathbf{s} * R\boldsymbol{\varepsilon} dx - \int_\Omega \boldsymbol{\sigma} * \boldsymbol{\varepsilon} dx \tag{22}$$

if and only if it is a solution of the finite-step nonlocal elastoplastic model coupled with damage. ■

The stationary conditions of the potential Π enforced at the point $(\boldsymbol{\varepsilon}, \mathbf{p}, R_i\boldsymbol{\alpha}_i, R\boldsymbol{\alpha}_5, \mathbf{s})$ with i = 1,…, 4, provides the finite-step nonlocal elastoplastic-damage model reported in Table 2 and Eqs. (14). In fact the stationary conditions are:

$$(0,0,0,0,0,0,0,0) \in \partial\Pi(\boldsymbol{\varepsilon}, \mathbf{p}, \boldsymbol{\alpha}_1, R\boldsymbol{\alpha}_2, R\boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, R\boldsymbol{\alpha}_5, \mathbf{s}) \tag{22}$$

which are equivalent to the following relations:

$$0 \in \partial_{\boldsymbol{\varepsilon}}\Pi \Leftrightarrow R\mathbf{s} + \boldsymbol{\sigma} = \mathrm{d}_{\boldsymbol{\varepsilon}-\mathbf{p}}\psi(\boldsymbol{\varepsilon} - \mathbf{p}, R_i\boldsymbol{\alpha}_i, R\boldsymbol{\alpha}_5) = \boldsymbol{\sigma}^e$$

$$(0,0,0,0,0,0) \in \partial_{(\mathbf{p},\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_5)}\Pi \Leftrightarrow \left\{ \begin{bmatrix} -\mathrm{d}_{\mathbf{p}}\psi \\ \mathrm{d}_{\boldsymbol{\alpha}_1}\psi \\ R\mathrm{d}_{\bar{\boldsymbol{\alpha}}_2}\psi \\ R\mathrm{d}_{\bar{\boldsymbol{\alpha}}_3}\psi \\ \mathrm{d}_{\boldsymbol{\alpha}_4}\psi \\ -R\mathrm{d}_{\bar{\boldsymbol{\alpha}}_5}\psi \end{bmatrix} \right. \in \partial D_1(\Delta\mathbf{p}, -\Delta\boldsymbol{\alpha}_i, \Delta\boldsymbol{\alpha}_5) \tag{23}$$

with i=1,…,4, where $\boldsymbol{\sigma}^e = -\mathrm{d}_{\mathbf{p}}\psi$, $\boldsymbol{\chi}_1 = \mathrm{d}_{\boldsymbol{\alpha}_1}\psi$, $\boldsymbol{\chi}_2 = \mathrm{d}_{\bar{\boldsymbol{\alpha}}_2}\psi$, $\boldsymbol{\chi}_3 = \mathrm{d}_{\bar{\boldsymbol{\alpha}}_3}\psi$, $\boldsymbol{\chi}_4 = \mathrm{d}_{\boldsymbol{\alpha}_4}\psi$, $\boldsymbol{\chi}_5 = -\mathrm{d}_{\bar{\boldsymbol{\alpha}}_5}\psi$. Further it results:

$$0 \in \partial_{\mathbf{s}}\Pi \Leftrightarrow R\boldsymbol{\varepsilon} \in \partial D_2(\Delta\mathbf{s}). \tag{24}$$

Reverting the steps above a solution of the finite-step nonlocal elastoplastic model with damage in the strain space makes the potential Π stationarity.

6. CONCLUSIONS

In this paper the systematic construction of a thermodynamic consistent integral nonlocal framework for an elastoplastic behaviour coupled with damage in the strain space is presented. Both plasticity and damage have a nonlocal behavior.

Then a variational formulation of the nonlocal elastoplastic and damage problem is derived. It is well-known that such formulations can serve as a basis for the numerical solution of initial boundary value problems in the sense of the finite element method.

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