The 2013 World Congress on
Advances in Structural Engineering and Mechanics (ASEM13)
Jeju, Korea, September 8-12, 2013

The achievable limits of operational modal analysis

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ABSTRACT

Operational modal analysis offers an economical way for identifying the modal properties (natural frequency, damping, mode shape) using output-only vibration data collected under working condition of the structure without artificial loading. As the modal properties are identified without using explicit information of the loading, they often have significantly higher uncertainty than those identified in free or forced vibration tests. It is of both scientific and engineering significance to understand the fundamental scaling law of their uncertainty, which governs the achievable limits of operational modal analysis. For example, how much data is needed for identifying the damping ratio to within 30% coefficient of variation? In a Bayesian probability context, this paper gives a fundamental answer for small damping, sufficient data duration and well-separated modes.

1. INTRODUCTION

Ambient vibration (output-only) tests have gained increasing popularity in both theory development and practical applications (Brincker 2001; Brownjohn 2003; Au 2011a). This is to a large extent attributed to its economy in implementation. Ambient vibration data are obtained when the structure is under unknown working load assumed to be random with broadband spectral characteristics. Ambient modal identification, often called „operational modal analysis', allows the extraction of modal properties under such context. In the absence of specific loading information, the uncertainty of the identified modal parameters is often significantly larger than that using forced vibration (known input) or free vibration tests where the signal-to-noise (s/n) ratio can be managed to an adequate level.

One question that arises frequently in performing ambient vibration tests is:

How much data do we need?

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The answer to question is by no means simple. In this paper we provide a fundamental answer to this question for well-separated modes under some asymptotic conditions, namely, small damping, high modal signal-to-noise ratio and sufficiently long data. These conditions are typically met in field testing applications of civil engineering structures and so the results can provide useful guidelines for performing ambient vibration tests in practice. In what follows, we shall first reason logically the context where the question and answer should be placed. We shall then outline the answer, referred as „uncertainty laws'. Practical implications shall also be provided.

2. BAYESIAN IDENTIFICATION FRAMEWORK

Suppose we have a digital record of acceleration time history data $\{\hat{\ddot{\mathbf{x}}}_j \in R^n : j = 1,...,N\}$ at n measured degrees of freedom (dofs) of a structure under „ambient condition'. From this data set we would like to identify the modal properties of a given mode of interest. The modal properties include primarily the natural frequency f, damping ratio ζ and (incomplete or partial) mode shape $\mathbf{\Phi}$ (an n-by-1 real vector). It is assumed that the structure is classically damped and the mode of interest is „well-separated' from other modes, in the sense that it dominates the frequency response (e.g., in terms of power spectral density, PSD) in its resonance band (see more later). It is also assumed that the data is of „good quality', in the sense that the contribution of modal response is large compared to the channel noise level in the resonance frequency band of the mode. „Ambient condition' or „broadband excitation' here refers specifically to the modal excitation having a constant PSD in the resonance frequency band. The modal excitation only need to be „locally white' within the resonance band, rather than the whole sampled frequency band. The ambient assumption here is much easier to justify that one might typically perceive.

In the presence of uncertainties or lack of information associated with the measurement (channel noise, finite data length) and modeling assumptions (e.g., classical damping, ambient excitation) one cannot expect to determine the modal properties exactly even in the presence of the data. The remaining uncertainty associated with the modal parameters given the data can be quantified fundamentally in a Bayesian identification perspective (Jaynes 2003; Beck 2012).

Let the modal parameters be collected in a vector $\boldsymbol{\theta}$ and let the measured data be denoted by D. In the presence of data all information about $\boldsymbol{\theta}$ is encapsulated in the „posterior distribution' $p(\boldsymbol{\theta} | D)$. According to Bayes' Theorem, it is given by

$$p(\boldsymbol{\theta} | D) = p(D | \boldsymbol{\theta}) p(\boldsymbol{\theta}) p(D)^{-1} \quad (1)$$

The RHS in (1) should be viewed as a probability distribution of $\boldsymbol{\theta}$ and so only its variation with respect to $\boldsymbol{\theta}$ matters. The last term on the RHS does not depend on $\boldsymbol{\theta}$ and so it is irrelevant to the knowledge of $\boldsymbol{\theta}$. The term $p(\boldsymbol{\theta})$ is the prior distribution we mentioned before. The term $p(D | \boldsymbol{\theta})$ is called the „likelihood function', which is the most important term because it dictates the mechanism by which the information in D can be utilized to infer $\boldsymbol{\theta}$. It can be (and must be) derived based on the identification model that relates $\boldsymbol{\theta}$ and D, which corresponds to a „forward' (rather than „inverse') problem.

Assume that the data is sufficient to narrow down the posterior distribution to having only a single peak, which is often valid in modal identification. In this case the uncertainty of each parameter in $\boldsymbol{\theta}$ can be conveniently characterized by the most probable value (MPV), which is where the posterior distribution is peaked; and a posterior standard deviation, which is related to the spread of the posterior distribution about its peak. For convenience an equivalent dimensionless measure called the posterior coefficient of variation (c.o.v.) is used, which is defined as the ratio of the posterior standard deviation to the MPV. Clearly, the posterior MPV and c.o.v. depend on the measured data. They can be obtained from the posterior distribution for given data. The process is primarily a computational problem which has been solved efficiently, typically in a matter of seconds (Au et al. 2013).

3. ASYMPTOTIC UNCERTAINTY LAWS

The foregoing discussion shows that the remaining uncertainty of the modal parameters can be quantified in terms of their posterior c.o.v.. This, however, does not provide much insight about how the posterior c.o.v. depends on various test configurations because it can only be calculated „point-wise' for a given set of data. The exact dependence of the posterior c.o.v. on test configurations and the measured data is expected to be extremely complicated and is unlikely to be described in a close-form explicit expression. However, a recent study (Au 2013a,b) shows that it is possible to obtain close-form expressions for the leading order term of the posterior c.o.v. for well-separated modes and under some asymptotic conditions, namely, small damping, high modal s/n ratio and sufficiently long data duration. The latter conditions are often encountered in ambient vibration tests of civil engineering structures and so the results can provide useful guidelines in practice. The results are referred as „uncertainty laws'. The derivation is quite lengthy (omitted here) but the results are remarkably simple. The full set of uncertainty laws also includes results on the mode shape, excitation intensity and channel noise but for practical purposes they are omitted in this paper. See Au (2013a, b) for a full story.

Let δ_f and δ_ζ denote respectively the posterior c.o.v. of the natural frequency and damping ratio. For a well-separated classically damped mode, small damping, high modal s/n ratio and sufficiently long data duration, it can be shown rigorously that („~' reads „asymptotic to' or „to the leading order')

$$\delta_f^2 \sim \frac{\zeta}{2\pi N_c B_f(\kappa)} \tag{2}$$

$$\delta_\zeta^2 \sim \frac{1}{2\pi\zeta N_c B_\zeta(\kappa)} \tag{3}$$

where f and ζ are respectively the natural frequency and damping ratio of the mode of interest;

$$N_c = T_d f \tag{4}$$

is a normalized data length equal to the duration of data T_d divided by the natural period (f^{-1}); and

$$B_f(\kappa) = \frac{2}{\pi}(\tan^{-1}\kappa - \frac{\kappa}{\kappa^2+1}) \tag{5}$$

$$B_\zeta(\kappa) = \frac{2}{\pi}\left[\tan^{-1}\kappa + \frac{\kappa}{\kappa^2+1} - \frac{2(\tan^{-1}\kappa)^2}{\kappa}\right] \tag{6}$$

are „data length factors' that are monotonic increasing function of the „bandwidth factor' κ. The latter reflects the amount of information that can be utilized in the data for identifying the mode of interest without incurring significant modeling error. More specifically, the uncertainty laws have been derived assuming that the mode is identified using information of the FFT of ambient vibration data within a frequency band of $f(1\pm\kappa\zeta)$.

4. PRACTICAL IMPLICATIONS

The uncertainty laws capture fundamentally the effect of test configurations and they have important implications on performing ambient vibration tests. First, δ_f^2 in Eq. (2) is proportional to ζ, while on the contrary δ_ζ^2 in Eq. (3) is inversely proportional to ζ. Ignoring the data length factors, $\delta_f^2/\delta_\zeta^2 = \zeta^2 << 1$. For small ζ encountered in applications, say, 0.5%~5%, this means that the damping ratio has much larger posterior uncertainty than the natural frequency or mode shape, and so it is likely to govern the required data length. This is consistent with common findings (Tamura et al. 1996; Au et al. 2012). The dependence of the uncertainty laws on ζ can be explained intuitively but it is omitted here; see Au (2013b).

Governed by the uncertainty of the damping ratio, the required data length as a multiple of the natural period to achieve a given posterior c.o.v. δ_ζ is given by, using Eq. (3),

$$N_c = [2\pi\zeta B_\zeta(\kappa)\delta_\zeta^2]^{-1} \tag{7}$$

To give a rule-of-thumb, consider a damping ratio of 1% and a bandwidth factor of $\kappa = 6$, which gives $B_\zeta(6) \sim 60\%$. The required data length is then $N_c \approx 27.5/\delta_\zeta^2$, say,

$$N_c \approx \frac{30}{\delta_\zeta^2} \quad (\zeta = 1\%, \kappa = 6) \tag{8}$$

This means that 300 natural periods are required to achieve a moderate posterior c.o.v. of $\delta_\zeta = 30\%$; 750 periods for $\delta_\zeta = 20\%$; and 3,000 periods for $\delta_\zeta = 10\%$. The corresponding c.o.v.s of the natural frequency are 0.67%, 0.27% and 0.067%, which are negligible. Smaller damping or bandwidth requires longer data length.

The value suggested in Eq. (8) is the minimum data length based on accuracy requirement and assuming good modal s/n ratio. In practice it will need to be traded off with other practical constraints. When little is known about the existence of a mode in a frequency band one may increase (e.g., double) the data duration to get a clearer picture of the spectrum for deciding the number of modes. On the other hand, there are situations that limit the data duration and hence the identification accuracy. Super-tall buildings (height >300m), for example, have a natural period in excess of 5 seconds. Assuming 1% damping, it requires over 4 hours to achieve a posterior c.o.v. of $\delta_\zeta = 10\%$. This duration is too long that significantly weakens the stationarity assumption in the stochastic load and the time invariance assumption of modal properties, giving rise to modeling errors that may invalidate the formulation. Wind loads during typhoons can change by orders of magnitude in a matter of an hour. The damping ratio can change significantly over such period as a result of amplitude dependence. In view of this, for super-tall buildings a c.o.v. of $\delta_\zeta = 30\%$ would be a reasonable accuracy to aim at, requiring about half an hour data. This may put a limit on the precision of field evidence for wind effects on long-period structures.

5. CONCLUSIONS

Uncertainty laws Eq. (2) and Eq. (3) have been presented for the natural frequency and damping ratio, respectively. They govern the achievable limits of accuracy in the natural frequency and damping ratio identified using ambient vibration data. They are derived fundamentally based on a Bayesian identification framework under asymptotic conditions of small damping and long data duration. Our discussion reveals clearly that the identification precision is primarily related to the spectral information in the data in the resonance band of the mode of interest. Information or complexities in other bands are irrelevant. As the Bayesian approach processes fundamentally the usable information in the data for given modeling assumptions, the uncertainty laws represent the lower limit of uncertainty that can be achieved by any method, including Bayesian and non-Bayesian methods. In the latter, uncertainty is interpreted as the ensemble variability of the estimates in a frequentist sense when there is no modeling error (Au 2012a).

6. ACKNOWLEDGMENTS

This paper is supported by General Research Fund 9041758 (CityU 110012) from the Research Grants Council of the Hong Kong Special Administrative Region, China.

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