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Structural Analysis of a Beam with Two Fixed Ends Using Screw Theory

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ABSTRACT

This paper presents an analytical modeling methodology for the stiffness analysis of a beam with two fixed ends. Beam modeling is a critical step in the design and analysis of beam structures and mechanisms. In this paper, we apply the Screw Theory to derive a symbolic model of the stiffness matrix with respect to the location of the loading and the geometry of the beam. Based on the model, the relationship between the loading location and the deformation is discussed. Finite Element (FE) model is built to verify the analytical model and the error is below 1 %. This proposed modeling method simplifies subsequent tasks such as design optimization and sensitivity study. The symbolic formulas can be the guidances for designers in doing beam structural analysis.

KEYWORDS: structural analysis, beam, screw theory, stiffness analysis, analytical model

1 INTRODUCTION

Beam analysis is one critical step in the analysis of structure and flexure mechanism Smith (2000), Shi and Su (2012), Shi et al. (2013). The derivation of a symbolic model Shi et al. (2014) is important in the analysis of the structural deformation under the loading of 6 degrees of freedom (DOF). By means of the Screw Theory, we can derive the analytical models of the beams. These symbolic formulas are easier for designers to recognize the geometric interpretation of each element in a compliance or stiffness matrix.

A lot of work has been done in the beam analysis using the Pseudo Rigid Body Model (PRBM). Howell and Midha (1995) built the parametric deflection approximation model for the compliant mechanisms, which is later developed to PRBM. Dado (2001) built a variable parametric pseudo-rigid-body model for beams with end loads. Yu et al. (2012) proposed a 2 R PRBM in the analysis of the beam deformation. Venkat and Su (2015) developed a 3-spring PRBM for a cantilever beam. Chen et al. (2011) described a 3 R PRBM model by using an improved particle swarm optimizer. However, PRBM is usually used in the deformation analysis of the beam less than 3DOF. By applying the Screw Theory, a 6DOF model can be derived for the beam analysis. Shi and Su (2013) derived a stiffness matrix for a beam with circular cross section. Shi et al. (2014) applied the Screw Theory in obtaining the analytical model of the beams with the different cross sections. Selig and Ding (2001) used the Screw Theory to derive the equation of a simple static beam. However, these previous work only focuses on the analysis of the cantilever beams, where the loading is located at one end of the beam. There is less work done in the beam modeling with a loading applied in the middle of a beam.

As shown in Fig. 1, a beam is suspended with two fixed ends. The concentrated forces and moments in the three directions are placed at one point of the beam. In this paper, we derive an analytical model of the stiffness matrix based on the loading location and geometrical parameters. The proposed modeling method is useful in beam modal analysis Xu et al. (2014), parametric sensitivity study and design optimization. The rest of the paper is organized as follows. Section 2 illustrates the Screw Theory and the stiffness matrices of the beams with a rectangular or circular cross section. Section 3 presents the modeling method of a beam with two fixed ends. In section 4, we provide an example to calculate the compliance matrices based on two different loading locations. In section 5, we analyze the compliance matrices based on the different loading locations. The Finite Element (FE) model is built to verify the analytical model. Section 6 is the conclusion of the paper.

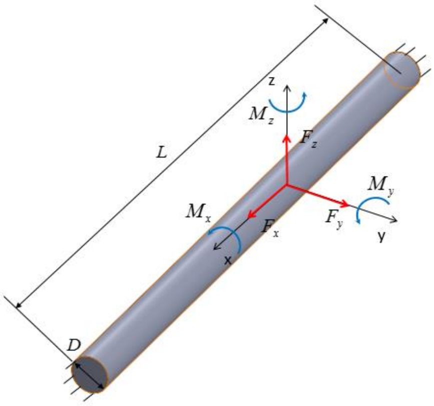

Figure 1.A beam with two fixed ends

2 SCREW THEORY

In this paper, we apply the Screw Theory in the derivation of the stiffness matrix and conduct the structural analysis. The applied loading in the Screw Theory is defined by a wrench vector $\hat{W} = (F_x, F_y, F_z, M_x, M_y, M_z)$ in the publication of Su et al. (2011). The deformation is defined by a general twist vector $\hat{T} = (\theta_x, \theta_y, \theta_z, \delta_x, \delta_y, \delta_z)$. They are related by,

$$\hat{W} = [K]\hat{T}, \quad \hat{T} = [C]\hat{W}, \quad [C][K] = [I], \tag{1}$$

where $[K]$ and $[C]$ are six by six stiffness and compliance matrices, respectively. The stiffness of a single beam with a rectangular cross section is

$$[K_b]=\frac{EI_z}{l}\begin{bmatrix} 0 & 0 & 0 & \frac{12}{l^2\eta} & 0 & 0 \\ 0 & 0 & -\frac{6}{l} & 0 & \frac{12}{l^2} & 0 \\ 0 & \frac{6}{l\kappa} & 0 & 0 & 0 & \frac{12}{l^2\kappa} \\ \chi\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{\kappa} & 0 & 0 & 0 & \frac{6}{l\kappa} \\ 0 & 0 & 4 & 0 & -\frac{6}{l} & 0 \end{bmatrix}, \tag{2}$$

where $\eta = t^2/l^2$, $\kappa = I_z/I_y = t^2/w^2$, $\chi = 1/2(1+\upsilon)$, υ is the Poisson's ratio and β is derived from

$$\beta = 12\left(\frac{1}{3} - 0.21\frac{t}{w}\left(1 - \frac{1}{12}\left(\frac{t}{w}\right)^4\right)\right). \tag{3}$$

The stiffness of a single beam with circular cross section is defined as

$$[K_w]=\frac{EI_z}{l}\begin{bmatrix} 0 & 0 & 0 & \frac{16}{l^2\eta} & 0 & 0 \\ 0 & 0 & -\frac{6}{l} & 0 & \frac{12}{l^2} & 0 \\ 0 & \frac{6}{l} & 0 & 0 & 0 & \frac{12}{l^2} \\ 2\chi & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & \frac{6}{l} \\ 0 & 0 & 4 & 0 & -\frac{6}{l} & 0 \end{bmatrix}, \tag{4}$$

where $\eta = d^2/l^2$.

3. MODELING OF A BEAM WITH TWO FIXED ENDS

In this section, we demonstrate the modeling method of a beam with two fixed ends, which has a circular cross section. The beam with a rectangular cross section follows the same derivation process. As shown in Fig. 2, a wrench is placed at one point on the center line of the beam. In the modeling of the structure, we firstly split the beam to two segments from the plane where the wrench is placed. The two segments with the same cross section are considered to be connected in parallel. The definition of serial and parallel connection of flexural structure is show in the publication Su et al. (2012), Shi (2013). Depending on the location of the separate plane, the two segments have different lengths. Secondly, by means of an adjoint transformation matrix [*Ad*], we can derive the stiffness matrix of the beam through the equation

$$[K]=[Ad_1][K_{w1}][Ad_1]^{-1}+[Ad_2][K_{w2}][Ad_2]^{-1}, \qquad (5)$$

where the subscript 1 means the left segment beam and 2 means the right segment beam. [K_{w1}] and [K_{w2}] are obtained by substituting the length *nL* and *L*, respectively. *n* is the length ratio of the left segment over the whole beam, which is corresponding to the location of the loading. [Ad_1] and [Ad_2] are the adjoint matrices and obtained by

$$[Ad]=\begin{bmatrix} R & 0 \\ DR & R \end{bmatrix}. \qquad (6)$$

Here, [*R*] is the rotation matrix. [*D*] is the skew-symmetric matrix defined by the translational vector d.

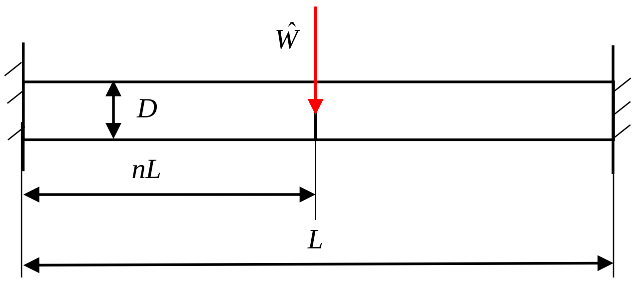

Figure 2.Modeling of a beam with two fixed ends

$$[R_1]=[Z(0)], d_1=(0,0,z), \qquad (7)$$

$$[R_2]=[Z(\pi)], d_2=(0,0,z). \qquad (8)$$

In this paper, the stiffness matrices of one single beam, Eq. (2) and Eq. (4) in Section 2 are based on the Euler Bernoulli Beam Theory. The geometric ratio of length over diameter should be over 10 so that the beam is considered as a long beam, where the tensile stress is dominant when comparing it with the shear stress. The derivation method of the beam with two fixed ends also needs to follow this rule. Thus, the proposed modeling method has some requirements. It is that the loading on the beam in longitudinal direction should be within the range shown as the center yellow range in Fig. 3. Within this location range, both of the two split segments are ensured to have the geometric ratio over 10.

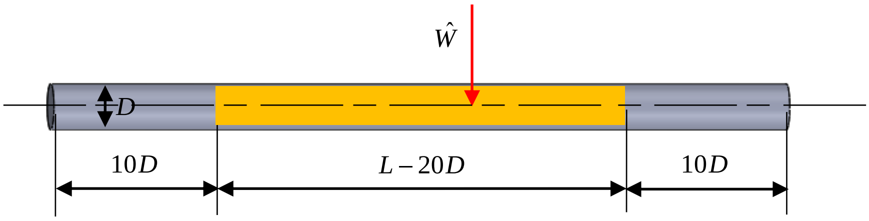

Figure 3.Loading requirement of the analytical model.

4. CASE STUDY

Here, we present an example of a beam with a circular cross section and it is made of tungsten. After substituting the material properties and the location ratio n_1 from Table. 1, we can obtain the compliance matrix at the location 1 as

$$[C_1]=\begin{bmatrix} 0 & 0 & 0 & 9.46E-8 & 0 & 0 \\ 0 & 0 & -8.73E-7 & 0 & 2.04E-8 & 0 \\ 0 & 8.73E-7 & 0 & 0 & 0 & 2.04E-8 \\ 1.82E-6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.49E-4 & 0 & 0 & 0 & 8.73E-7 \\ 0 & 0 & 3.49E-4 & 0 & -8.73E-7 & 0 \end{bmatrix}. \quad (9)$$

Here, the units of the rotational and translational displacements are radian and millimeter (mm), respectively. The units of the forces and moments are newton (N) and newton-millimeter (N·mm), respectively. For example, the value 3.49E-4 mm/N in column 2 of Eq. (9) means the translational displacement in the y direction caused by 1 N force in the y direction. The value 2.04E-8 radians/N·mm in column 6 of Eq. (9) means the rotational displacement in the z direction caused by the moment, 1 N·mm in the z direction. After substituting the value of $n_{2,}$ we can also obtain the compliance matrix of the loading at the location 2.

$$[C_2]=\begin{bmatrix} 0 & 0 & 0 & 9.85E-8 & 0 & 0 \\ 0 & 0 & -2.23E-19 & 0 & 1.89E-8 & 0 \\ 0 & 2.23E-19 & 0 & 0 & 0 & 1.89E-8 \\ 1.89E-6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.95E-4 & 0 & 0 & 0 & 2.23E-19 \\ 0 & 0 & 3.95E-4 & 0 & -2.23E-19 & 0 \end{bmatrix}. \quad (10)$$

Table 1. Parameters for case study

Beam length (L)	500 mm	Yong's Modulus (E)	210 GPa	Location value (n_1)	0.4
Diameter (D)	20 mm	Poisson's ratio (υ)	0.3	Location value (n_2)	0.5

5. DISCUSSION OF THE MODELING

From Eq. (9) and (10), we can find the differences between the compliance according to the different locations. Some conclusions are obtained as following:

1) The absolute values of the elements in Column 2 and 3, Column 5 and 6 are respectively the same. The reason is that the y and z directions are towards the radial direction, and the circular cross section is symmetrical. If the beam has a rectangular cross section, the absolute values of these elements will be different.
2) The value at column 5 and 6 is -8.73E-7 in [C_1] but 2.23E-19 in [C_2]. This is because this value means the translational displacement in the y or z direction caused by the moment in the y or z direction, respectively. At location 2, the loading is at center in the longitudinal direction of the beam. Thus, the pure moment theoretically does not cause the translational displacement. 2.23E-19 is the calculation precision of the software and it can be considered as 0.
3) The value at column 2 and 3 is -8.73E-7 in [C_1] but 2.23E-19 in [C_2]. This is because this element value means the rotational displacement in the y or z direction caused by the force in the y or z direction, respectively. At the location 2, the loading is at center in longitudinal direction of the beam. Thus, the pure moment theoretically does not cause the translational displacement.

In order to evaluate the derived stiffness matrix in Eq. (9), we build a Finite Element (FE) model to analyze the model and calculate the errors of the analytical model. As shown in Fig. 4, the beam is placed at the coordinate system according the Fig. 1 and with the parameters shown in Table 1. Concentrated force is located by means of the location value *n*, 0.4. The model is well meshed with multiple layers.

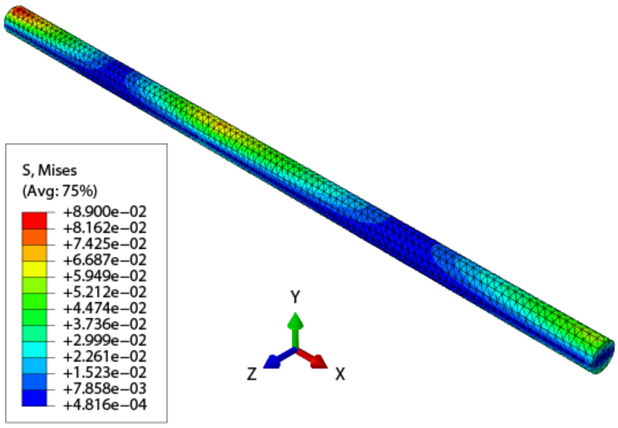

Figure 4. Finite Element model.

As shown in Fig. 4, we apply a force of 1 N in the y direction on the beam. The corresponding translational and rotational displacements at the point of the loading are shown in Table 2. When we multiply a wrist (0,1,0,0,0,0) to Eq. (9) , we can also obtain the rotational displacement in the z direction as 8.73E-7 radians and the translational displacement in the y direction as 3.49E-4 mm. When comparing the values with the FE model, we can calculate the errors of the analytical model. They are 0.57 % for the rotational displacement and 0.8 % for the translational displacement.

Table 2. Parameters for case study

$\theta_x(radian)$	$\theta_y(radian)$	$\theta_z(radian)$	$\delta_x(mm)$	$\delta_y(mm)$	$\delta_z(mm)$
-1.1039E-10	-1.3871E-11	8.65617E-7	7.0935E-10	3.5050E-4	8.9379E-9

6. CONCLUSION

In this paper, an analytical modeling method based on the Screw Theory is presented in deriving the stiffness matrix of a beam with two fixed ends. Based on the model, the relationship between the displacements in 6DOF and loading location is analyzed. FE model is built for verifying the analytical modeling method and it shows that the error is below 1 %. The benefits of these symbolic formulas are that designers can easily conduct structural analysis of the beam, and they simplify subsequent tasks such as design synthesis and sensitivity analysis. This modeling method can also be adopted for the analysis of parallel mechanism.

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