

Dynamic Buckling Study of Laminated Composite Stiffened Plate

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ABSTRACT

The paper deals with dynamic buckling studies of thin-walled laminated composite stiffened plate subjected to impulse axial loads along the edge, with different loading functions and durations.. The approach adopted is based on the equations of motion, which are numerically solved using a finite element code (ABAQUS/Explicit). The S4R shell element is adopted in the present analysis to model the plate and the stiffeners. The effect of loading functions, loading durations and stiffener type on the dynamic buckling behaviour of laminated composite stiffened plate are considered in the present study. The corresponding dynamic buckling loads are related to the static buckling loads.

Keywords: Dynamic buckling, stiffened plate, laminated composite, Dynamic Explicit analysis, Finite element, ABAQUS

1. INTRODUCTION

With the increased application of fiber reinforced composites in various fields, research on their behaviour for different structural form has also increased. Most commonly used structural forms are plates, used in aircraft, ship and automotive industries. The performance, i.e. strength/stiffness to weight ratio of the plates is enhanced by adopting suitable stiffened forms. In many practical situations these stiffened plates are subjected to in-plane compressive dynamic loads. The dynamic nature of the load are very random in nature. The in-plane compressive dynamic loads on the edges of the stiffened plates are coming from the adjacent structures. In these situations, buckling of these thin-walled stiffened plate becomes one of the prominent phenomenon for failure. Buckling of the plate may occur due to these dynamic loads subject along the edges. In many cases the dynamic buckling loads are less than the static buckling load. From, the point of design, if some plate is capable to sustain some particular amount of static in-plane load before buckling may buckle earlier with dynamic load of less magnitude. The nature and duration of the dynamic load has significant effect on the buckling behaviour. So, a designer must consider the dynamic nature of the load while designing such structural components.

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The literature dealing with the dynamic buckling behaviour of laminated composite stiffened/un-stiffened plates is few. One of the first researchers to investigate dynamic buckling was Zizicas (1952), who developed a theoretical solution for the case of a simply supported rectangular plate under time-dependent in-plane loads although a buckling criterion was not determined. Budiansky and Roth (1962) studied, in the 60's, the axisymmetric dynamic buckling of clamped shallow spherical shells and calculated the deformation of the shell due to a transient pressure loading. They also suggested a criterion for dynamic buckling as a function of loading duration. Weller et. al. (1981) calculated the Dynamic Load Amplification Factor (DLF) for a series of beams and plates under differing pulse durations and geometric imperfection using the ADINA computer code. Lindberg and Florence (1987) described the effect of high amplitude short duration load in all problems they have taken. Their published work emphasizes that the critical dynamic buckling loads of long duration (e.g., step load) can be smaller in magnitude than the corresponding static buckling load. Ari-Gur and Simonetta (1997) carried out the investigation on dynamic pulse buckling of rectangular composite plates through an explicit finite-difference integration scheme. The applied load was either a force or displacement pulse, and the buckling loads were determined for various loading durations and material lay-up configurations. Petry and Fahlbush (2000) investigated the dynamic stability behavior of imperfect simply supported plates subjected to in-plane pulse loading. For the calculation of dynamic buckling loads a stress failure criterion was applied. The influences of pulse duration, shock function, imperfection, geometric dimensions and limit stress of the material were discussed. Yaffe and Abramovich (2003) used the finite element code ADINA to investigate the buckling behavior of aluminium stringer stiffened shells under axial dynamic loading. They showed numerically that the shape of the loading period, as well as the initial geometric imperfections had a great influence on the dynamic buckling of the shells. Bisagni (2005) reported the results of the dynamic buckling due to impulsive loading of thin-walled carbon fiber reinforced plastics shell structures under axial compression. The approach adopted was based on the equations of motion, which were numerically solved using the finite element code ABAQUS/Explicit. It was shown numerically that the initial geometric imperfections as well as the duration of the loading period had a great influence on the dynamic buckling of the shells. Featherston et.al. (2010) studied an intermediate velocity impact by carrying preliminary tests to find the feasibility of using high speed digital image correlation (DIC) and hence validated results obtained by FEA. Uniaxial compression was applied on a longitudinally stiffened panel specimen, clamped within a rig designed to provide built-in end conditions and allow motion of one end in the direction of loading only. The specimen was tested using an accelerated drop test rig and full field displacement contours are obtained using a high speed DIC system. Eglitis (2011) confirmed the high sensitivity of cylindrical shells to the geometrical imperfections and demonstrated that the imperfection shape has greater influence on the buckling load than its magnitude. He recommended to perform the geometric imperfection measurements on real shell specimens in order to model the buckling of a shell accurately and obtain reliable results.

Some research works on dynamic buckling behaviour of composite structures with and without stiffeners with step loads with different duration are available in the literature. However, there is limited research on effect of various types of load functions

on the dynamic buckling behaviour of the laminated composite stiffened plates. The present investigation is devoted to find the dynamic buckling characteristics of a laminated composite stiffened plate subjected to axial impulsive loading with different load function and different durations numerically. Dynamic buckling analysis is done by adopting equations of motion approach and numerical equations are solved using Finite element analysis based software ABAQUS explicit. In the present study Budiansky-Roth criterion (1962) is considered. Influence of different loading duration and loading function on the instability behaviour are shown numerically. The results reported are purely numerical. Experiments are very useful to have clear and better understanding.

2. THEOTRICAL FORMULATIONS

The definition of dynamic buckling is arbitrary and consequently there is no unique criterion as yet for determination of dynamic buckling, nor do guidelines for design of dynamic buckling resistant structures exist. Budiansky and Hutchinson (1966) proposed a criterion that leads itself to a rational definition of dynamic buckling.

A widely accepted definition of dynamic buckling of imperfect structures states that buckling occurs when a small increase in the load intensity results in an unbounded growth of the deflections Ekstrom(1973). The three different methods are usually considered for the evaluation of the critical conditions for dynamically loaded elastic systems: total energy phase-plane approach, total potential energy approach and equation of motion approach, often called also Budiansky-Roth criterion (1962). According to the Budiansky-Roth criterion, the equations of motion are solved for various load parameters, such as magnitude and duration, obtaining the system responses. The critical condition is then defined when a large change in response is obtained. There are five types of buckling criteria based on response considered:

CRITERIA I (Ari-Gur and Simonetta (1997)): It relates the peak lateral deflection to the pulse intensity. Buckling occurs when, for a given pulse shape and duration, a small increase in the pulse intensity causes a sharp increase in the rate of growth of the peak lateral deflection.

CRITERIA II (Ari-Gur and Simonetta (1997)): It associates dynamic buckling with a pattern of short wavelength deflection shape. Buckling occurs when a small increase in the pulse intensity causes a decrease in the peak lateral deflection. This criterion is relevant to impulsive loads only and may be used to complement the first criterion.

CRITERIA III (Ari-Gurand and Simonetta (1997)): It applies to a force pulse Buckling occurs when a small increase in the force intensity causes a sharp increase in the peak longitudinal displacement at loading edge. It occurs because the structural resistance to the in-plane compression diminishes when the dynamic lateral deflections grow rapidly. This criterion in referred as force criterion.

CRITERIA IV (Ari-Gur and Simonetta (1997)): It applies to a displacement pulse. Buckling occurs when a small increase in the pulse displacement intensity causes a transition of peak reaction force from compression to tension.

CRITERIA V (Volmir (1958)): It states that, if the deformations caused by the static buckling load are possible by any combination of magnitude and duration of

dynamic load, then that dynamic load is regarded as dynamic buckling load, also called as Volmir criterion. It is defined as: "Dynamic critical load corresponds to the amplitude of pulse load of constant duration at which the maximum plate deflection is equal to some constant value".

In the present investigation, Budiansky-Roth(1962) criterion (CRITERIA III) is considered. The dynamic buckling analysis of the laminated composite plate is carried out using the finite element software, ABAQUS. ABAQUS is a general purpose finite element program with linear static and dynamic analysis capabilities. At first, buckling analysis of the plate subjected to static axial compression is performed by linear static (Eigen value) analysis using ABAQUS/Standard. The dynamic buckling analysis of the stiffened plate under impulsive loading is then carried out by explicit integration scheme used in ABAQUS/Explicit. In the dynamic analysis, the basic equations of motion of a composite laminated plate are

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F(t)\}$$

Where $[M]$ is the mass matrix, $[C]$ the damping matrix, $[K]$ the stiffness matrix including non-linearity, $\{u\}$ the nodal displacement vector, $\{\dot{u}\}$ the nodal velocity vector, $\{\ddot{u}\}$ the nodal acceleration vector and $\{F(t)\}$ the load vector.

The equations of motion are solved for various values of the loading and the value at which displacement reaches displacement under static load is considered critical. When monitoring the system response through displacements of selected points for small values of the loading parameter, small oscillations are observed, the amplitudes of which gradually increases as the loading is increased. Implementation of this criterion requires to solve the equations of motion for different values of the loading parameter and then the displacement amplitude versus loading curve is plotted from which the critical loading value is determined.

For impact loading, the load of specific magnitude is suddenly applied with finite duration. Different values of load duration, greater than and less than the natural period of the shell are considered for analysing the plate. The dynamic buckling loads thus obtained are compared with the static buckling loads.

3. RESULTS AND DISCUSSION

Various problems are taken under consideration in order to validate the results obtained by different research papers. Then the dynamic analysis of the laminated composite plate is carried.

3.1 Convergence and Validation

The convergence and validation study is presented first and the system for the present investigation is presented. The convergence study is not reported exclusively. The different examples considered are,

3.1.1 Buckling of a Composite Stiffened Cylinder

The laminated composite stiffened shell(Fig.1) is characterized by an inner radius of 350 mm. The overall length is equal to 540 mm. The skin is made of two plies oriented at $[45^\circ/-45^\circ]$, where zero degree is the axial direction of the shell, and the numbering of the plies starts from the inner side. Eight L-shaped stringers equally spaced in the circumferential direction are bonded inside skin. The blade of the stiffeners is 25 mm deep, while the flange attached to the skin is 32 mm wide. The stringers consist of 12 plies oriented at $[0^\circ/90^\circ]_{3s}$. In correspondence of the stringers, on the outer side of the shell, three plies of reinforcement are added with $[0^\circ/45^\circ/-45^\circ]$ orientation. The reinforcement is 40 mm width. In the model the circumferential to longitudinal divisions of the cylinder skin is 320×80 , and the whole model consists of 39680 S4R shell elements. The dimensions of the elements are 4×6.75 mm for the reinforcement, the skin under the reinforcement and the stiffener flange, 6.25×6.75 mm for the stiffener blade and 7.83×6.75 mm for the other parts of the skin. The material properties are reported in Table.1.

All six boundary conditions are restrained in the bottom surface of the shell, while in the top surface the axial displacement is allowed keeping restrained the other five boundary conditions. The static buckling load is reported in Table.2 along with the result of Patel et.al. (2011). It is observed from Table 2 that the results are matching well. The small difference in result may be due to the different mesh meshes in both analysis.

Table 1: Mechanical properties of the CFRP ply

Elastic modulus E_{11} (MPa)	57765
Elastic modulus E_{22} (MPa)	53686
Shear modulus G_{12} (MPa)	3065
Poisson's ratio ν_{12}	0.048
Density (kg/m^3)	1510
Thickness (mm)	0.33

A uniform global element size of 0.005m square was taken for present analysis thus creating 47520 elements on the skin, 6048 elements on the reinforcement and 8640 elements on the stringers.

Table 2: Validation of static buckling load

ABAQUS/Standard: eigenvalue analysis	Buckling load of Cylinder $[45^\circ/-45^\circ]$ in (KN)
Patel et.al. (2011)	245
Present	236

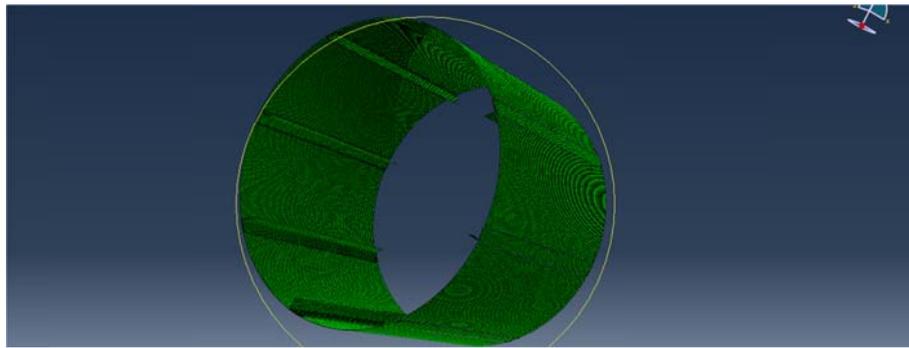


Fig 1: Stiffened Cylinder

Based on the convergence and validation study the mesh size of 3mm and 1.5mm is taken for the stiffened plate skin and stiffener in the subsequent analysis.

3.1.2 Buckling of simply supported Square plate

A square, simply supported, uniformly loaded angle-ply ($\theta^\circ/-\theta^\circ/\theta^\circ$) plate with $E_{11}=60.7$, $E_{22}=24.8$, $G_{12}=12.0$, $\nu_{12}=0.23$, $a/h=100$ is considered for validation purpose . The non-dimensional buckling load is $P_{nd} = P_{cr}b^2/D_0$, $D_0=E_{11}h^3/12(1-\nu_{12}\nu_{21})$. The non-dimensional buckling load is shown in Table.3 along with the finite element results of Patel et.al. (2003) and Bisagni (1998). It is observed from Table 3 that the results are matching well.

Table 3: Comparison of non-dimensional buckling loads for a square, simply supported, uniformly loaded angle-ply ($\theta^\circ/-\theta^\circ/\theta^\circ$) plate.

Angle in degree	Present	Patel et.al. (2003)	Bisagni (1998)
0	23.39	23.3769	23.3778
15	24.04	24.0048	23.9906
30	25.41	25.3317	25.2778
45	26.1	26.0246	25.9514

3.2 Laminated composite stiffened plate under investigation

Laminated composite stiffened plates with skin of four plies are used in the analysis. Plates having plan dimensions as length (L) of 600 mm, width (B) of 400 mm with a thickness (t) of 1 mm is used in the present study (Fig.2). Layup sequence, $[0^\circ/45^\circ/-45^\circ/0^\circ]$ was considered, where the 0° corresponds to the longitudinal direction of the plate. Each ply is 0.25 mm thick and has the material properties are specified in the Table 4.

Table 4: Mechanical properties of the laminate

Property	Value
Longitudinal Tensile Modulus E_{11} (MPa)	52,000
Transverse Tensile Modulus E_{22} (MPa)	52,000
In-plane Shear Modulus G_{12} (MPa)	2350
Poisson's ratio ν_{12}	0.302
Density (kg/m^3)	1320
Thickness of Ply (mm)	0.25

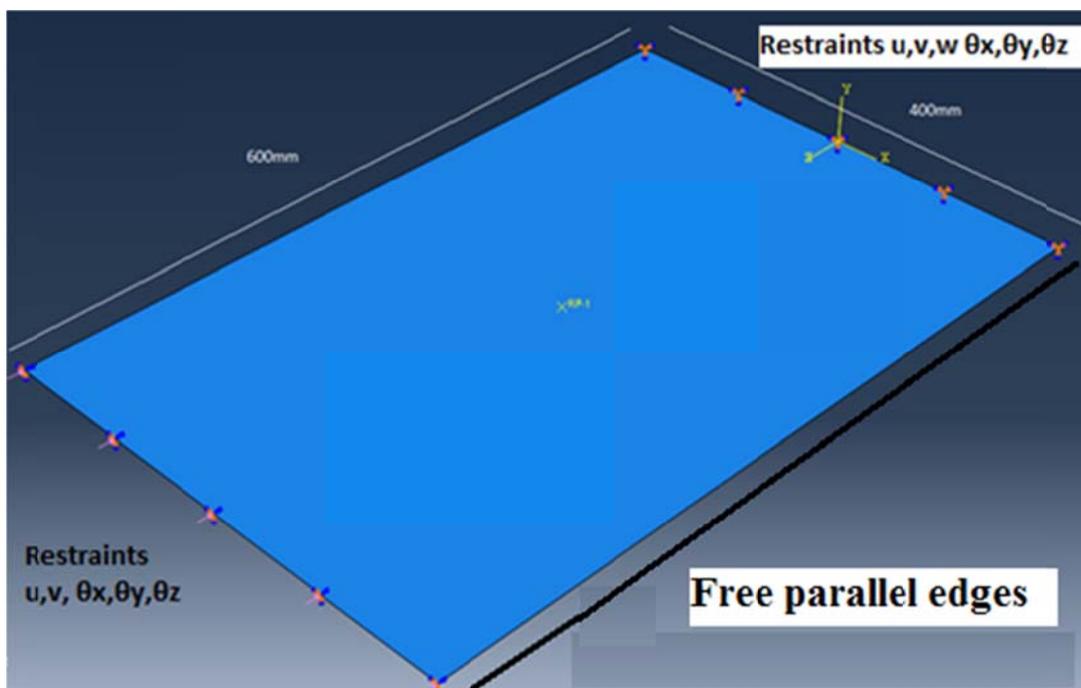


Fig. 2 Geometry and boundary conditions of laminated composite plate

The stacking sequence of plies in a laminated plate influences the coupling properties and consequently the buckling strength of the laminate. Symmetric laminates are used in order to avoid coupling properties. Angle-ply laminates offer more shear stiffness than normal and cross ply laminates due to which they find more applications over other laminate configurations. $[0^\circ/45^\circ/-45^\circ/0^\circ]$ is selected for the current study. Stiffened plates have T-shaped stiffeners of same material and lamination scheme with depth of stiffener web 10 mm and width of flange is 20 mm. The flange and web of the T-stiffeners are of same thickness and lamination scheme with the plate skin. In one case stiffener without cutouts(Fig. 3a) and other case stiffener with cutouts(Fig. 3b) in the web portion are considered. The cutouts in the web portion are made to make it castellated. The connection is bolts. The fixed edge has no

u , v , w , θ_x , θ_y , and θ_z deformations. The loading edge has only w allowed to deform. The effect of stiffener on the strength of the plate against stability is studied in this paper. The stiffeners were monolithically casted like modelled so there is no difference in the strains at the junction of the plate and stiffener. 2 stiffeners were kept at a distance of 120mm from central axis on either side. The connections were end to end along length. The choice of number of bolts was just to not allow separation in the least number.

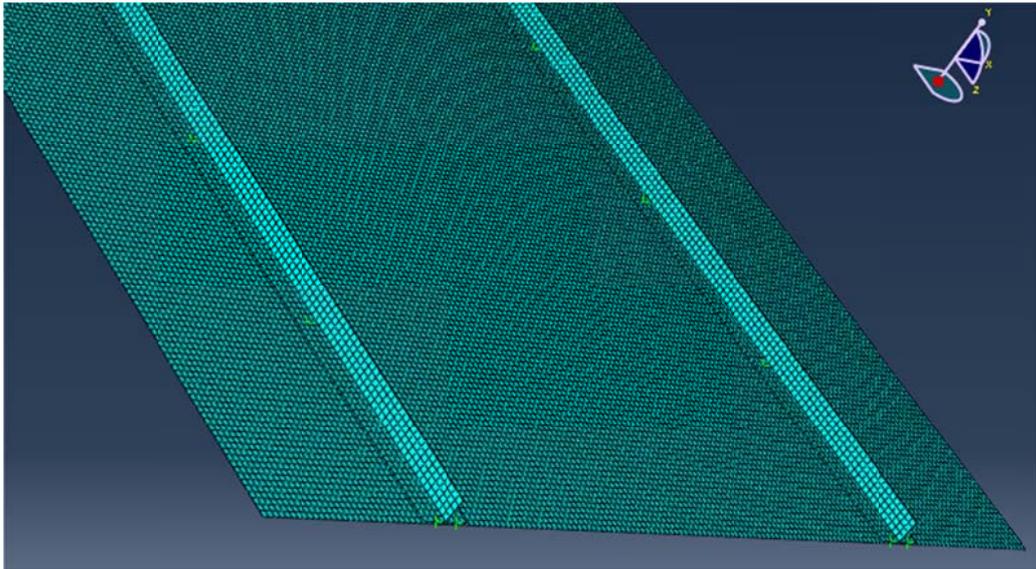


Fig. 3a Geometry of laminated composite T-Stiffened plate

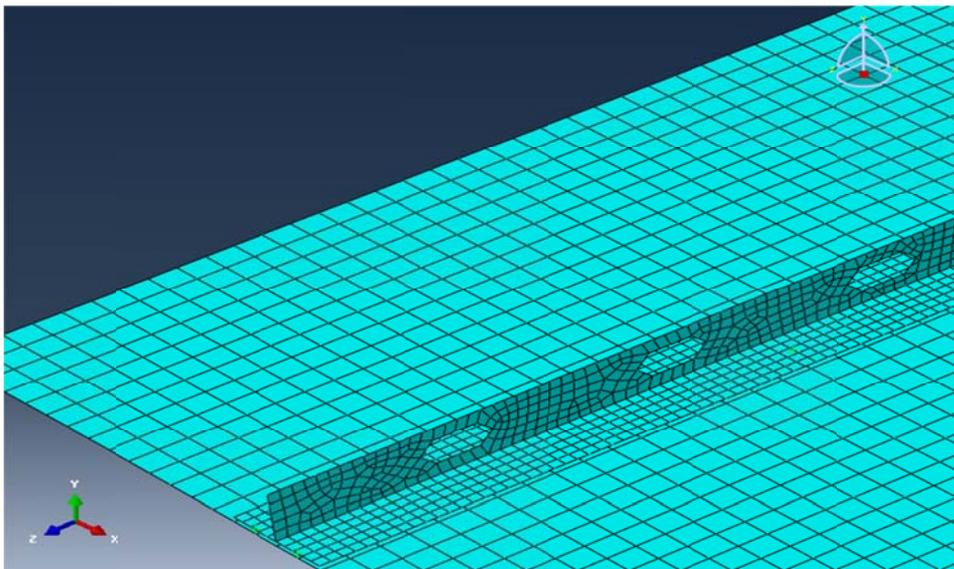


Fig. 3b Geometry of laminated composite castellated Stiffened plate

The stiffeners are of 2 types, castellated and ordinary T-beam. They are connected by 8 bolts from end to end along the 600mm length. The dynamic loads are applied using various functions as Instantaneous, $\sin(\mu t)$, $0.5 \cdot (1 + \sin(\mu t))$, $\frac{1}{\sqrt{2}} (\cos(\mu t) + \sin(\mu t))$ and Ramp functions as shown in Fig.5. Figure 4 shows the amplitude option of loading in ABAQUS/Explicit.

The amplitude of the above functions is 1, and thence the load value provided will be the applied maximum load. The general formula followed in the Amplitude option of ABAQUS is simplified for the above functions as follows.

$$F(t) = L * (A_0 + A_1 \cos(\mu t) + B_1 \sin(\mu t))$$

A_0 = Initial Amplitude, A_1 and B_1 are the Fourier series constants, L is the applied load on the shell edge, and initial time was kept at zero.

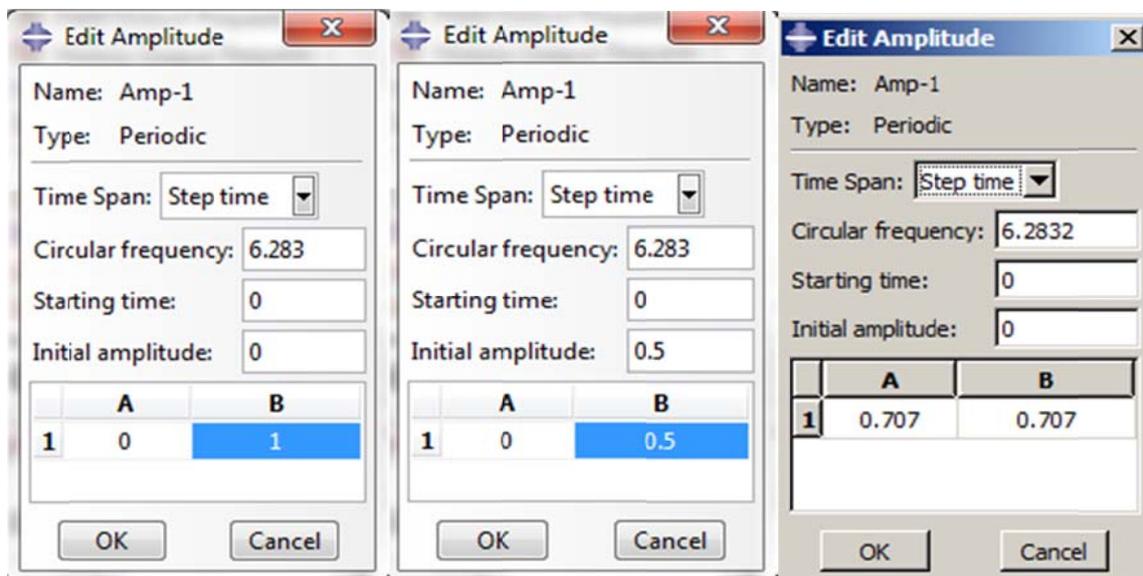


Fig 4 Amplitudes entered in ABAQUS

The value of μ is the frequency in every step such that the positive quarter cycle is exactly completed at the end of the step. This is to ensure that the plate undergoes only axial compression force. Therefore, $\mu = \pi/2T$. In the case of stiffened plates the ramp and $\sin(\mu t)$ functions were not effective in dynamically buckling as desired. The t takes the values of $0, 0.2T, 0.4T, 0.6T, 0.8T$ and T .

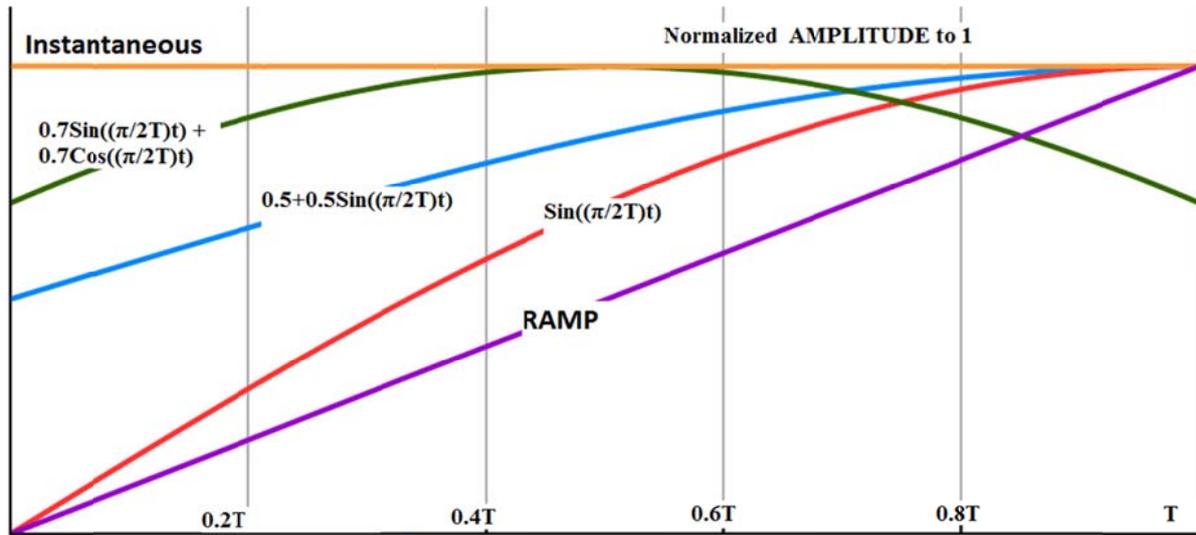


Fig 5: Loading functions Plots

The load in the stiffened plate is applied along the one edge of 400mm. The opposite 400mm edge is fixed (Fig.2). In the loading edge the axial deformation (w -in this case) is only allowed and all other five DOF are restricted. The 600mm edges are free. The load is applied in skin of the stiffened plate only. The static buckling load for stiffened plate in 1st case (stiffener without cut-out, Fig.3a) is 2766N/m and in 2nd case(stiffener with cut-out, Fig.3b) is 2507N/m. After that the dynamic analysis with different loading function and different durations are performed. That particular combination of loading function with duration is called as dynamic buckling load which shows excessive deformation. The results are shown in Table 6.

Table 6: Dynamic buckling Load (N/m) along the edge with varying loading function

Type of Stiffened plate	Type of loading function			
	Instantaneous	$\text{Sin}(\mu t)$	$0.5 \cdot (1 + \text{Sin}(\mu t))$	$\frac{1}{\sqrt{2}}(\text{Cos}(\mu t) + \text{Sin}(\mu t))$
Case-1	(Greater than Static buckling load) 3500@ T=0.5s 3100@ T= 1s	(Greater than Static buckling load) 3245 @ T=0.5s 2946 @ T= 1s	(less than Static buckling load) 2356@ T=0.5s 2389@ T= 0.25s	(less than Static buckling load) 2343@ T=0.5s 2402@ T= 0.25s
Case-2	(Greater than Static buckling load) 3230@ T=0.5s 2870@ T= 1s	(Greater than Static buckling load) 3018 @ T=0.5s 2891 @ T= 1s	(less than Static buckling load) 2306@ T=0.5s 2389@ T= 0.25s	(less than Static buckling load) 2343@ T=0.5s 2402@ T= 0.25s

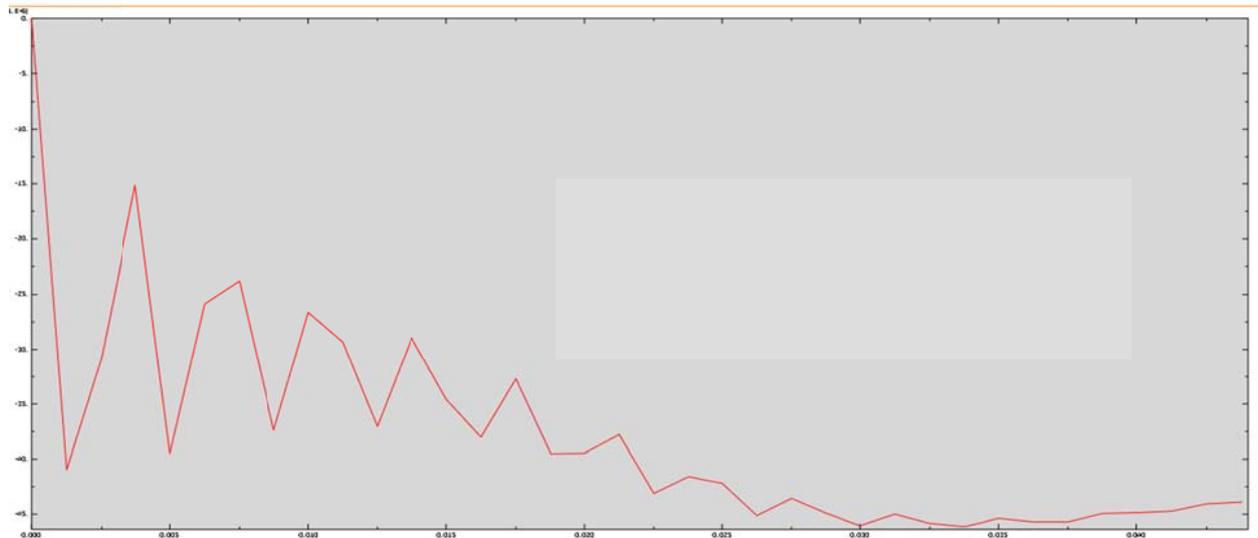


Fig 6: Axial displacement behaviour of T-stiffened plate for $2500\text{Sin}\mu t$ @ $T=0.5\text{s}$

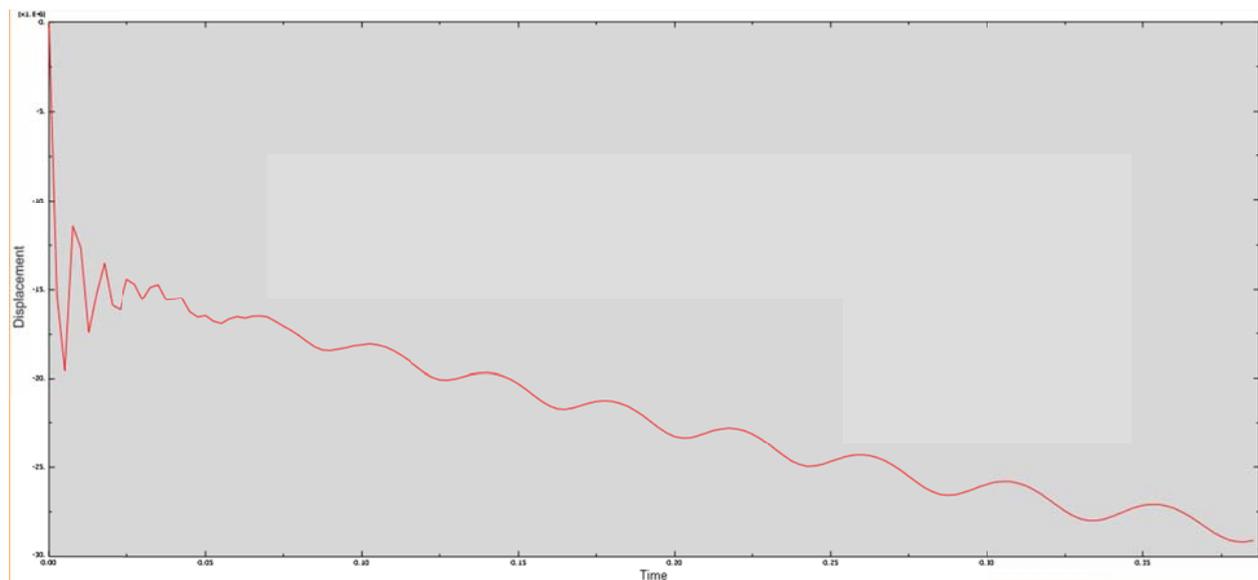


Fig 7: Axial displacement behaviour of Castellated plate for $2500\text{Sin}\mu t$ @ $T=0.5\text{s}$

The above figures (Fig 6 and 7) have not shown sudden excessive axial deformation and hence not been categorized as dynamically buckled.

The following figures are considered buckled as per sudden increase in deformation axially. The deformed geometry of the stiffened plate is also shown in parallel with the axial deformation plots. The figures are only for castellated stiffened plate. The buckled plates were obtained on the application of sinusoidal loads of function, $0.5*(1+\text{Sin}(\mu t))$ and $\frac{1}{\sqrt{2}}(\text{Cos}(\mu t)+\text{Sin}(\mu t))$. The time periods are 0.25 and 0.5 seconds. These functions are

plotted in Fig.8. Figure 9 to Fig. 12 shows the deformed plate and sudden change of deformation for four different cases.

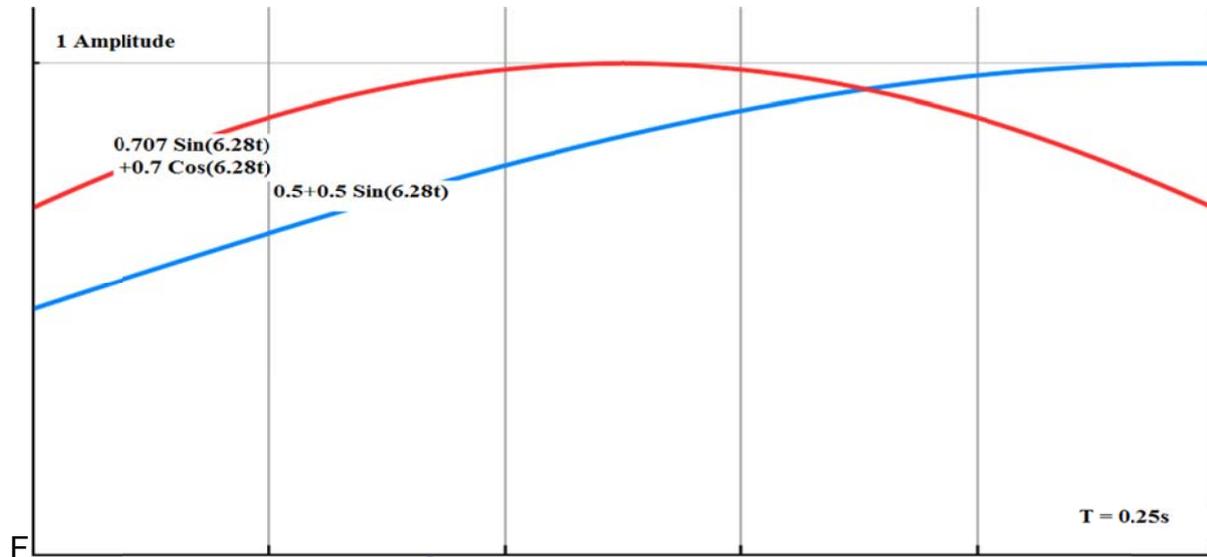


Fig 8: loading function plots @ T=0.25s

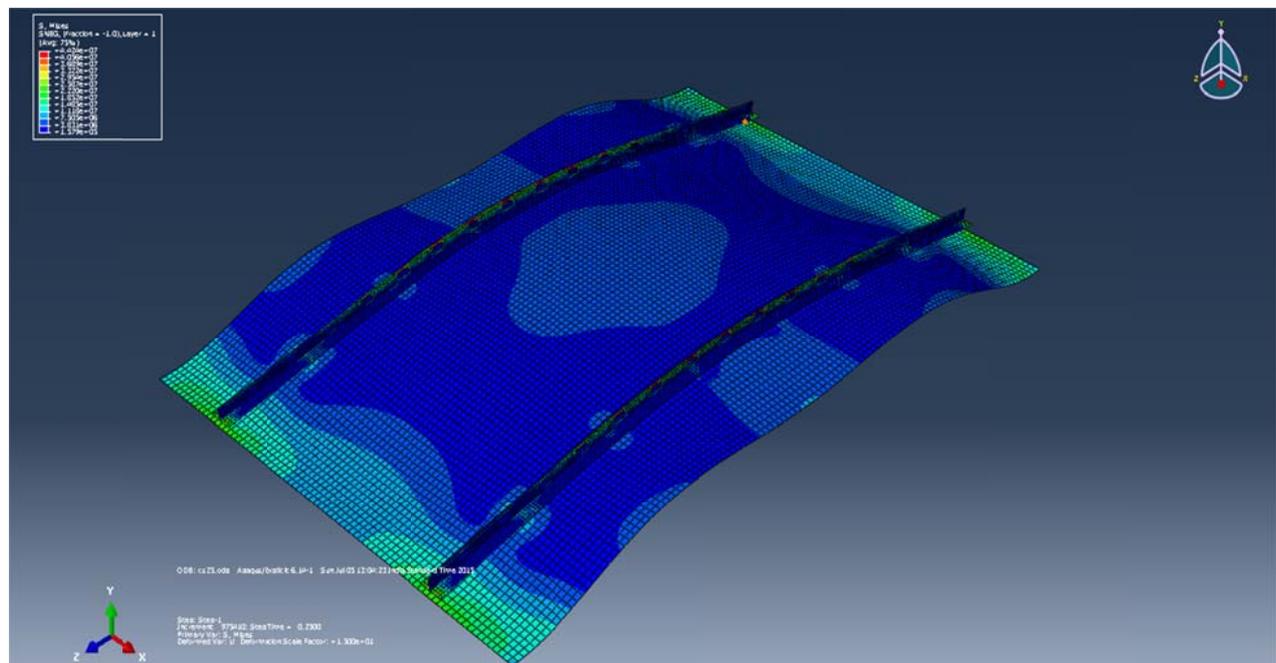


Fig. 9a Deformed stiffened Castellated plate for $(2401/1.414) \times (\cos 2\mu t + \sin 2\mu t)$ @ $T=0.25s$

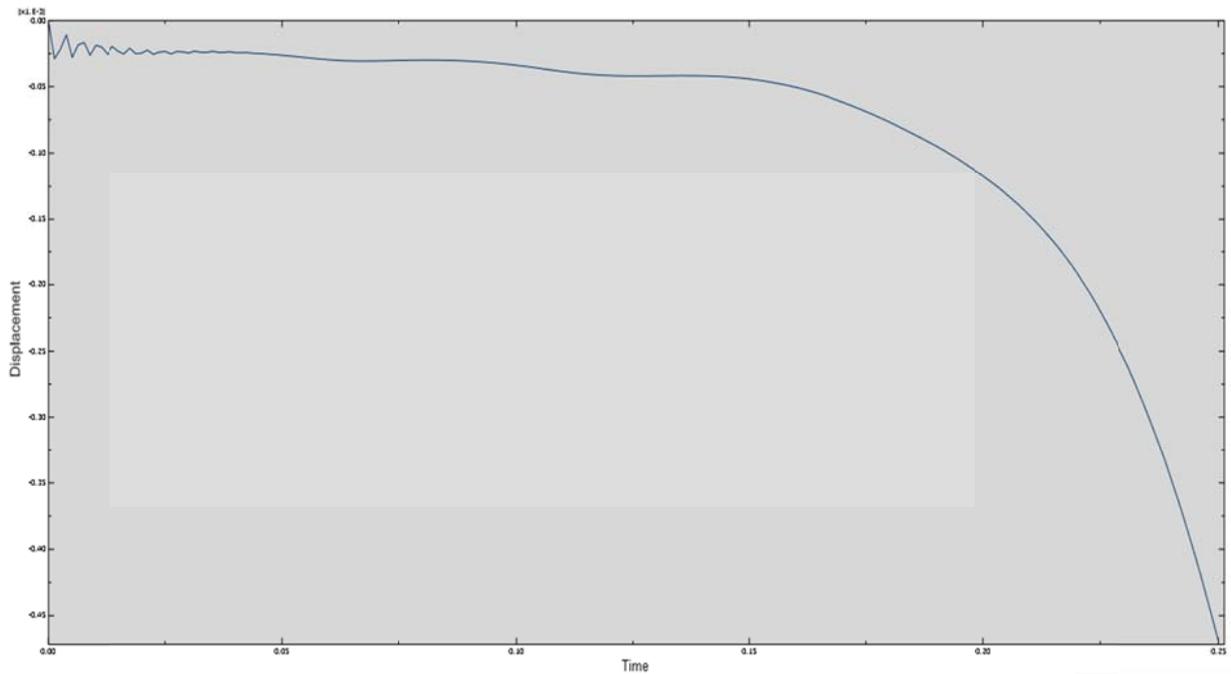


Fig 10b: Axial displacement behaviour of Castellated plate for $2389x(0.5+0.5\sin 2\mu t)$ @ $T=0.25s$

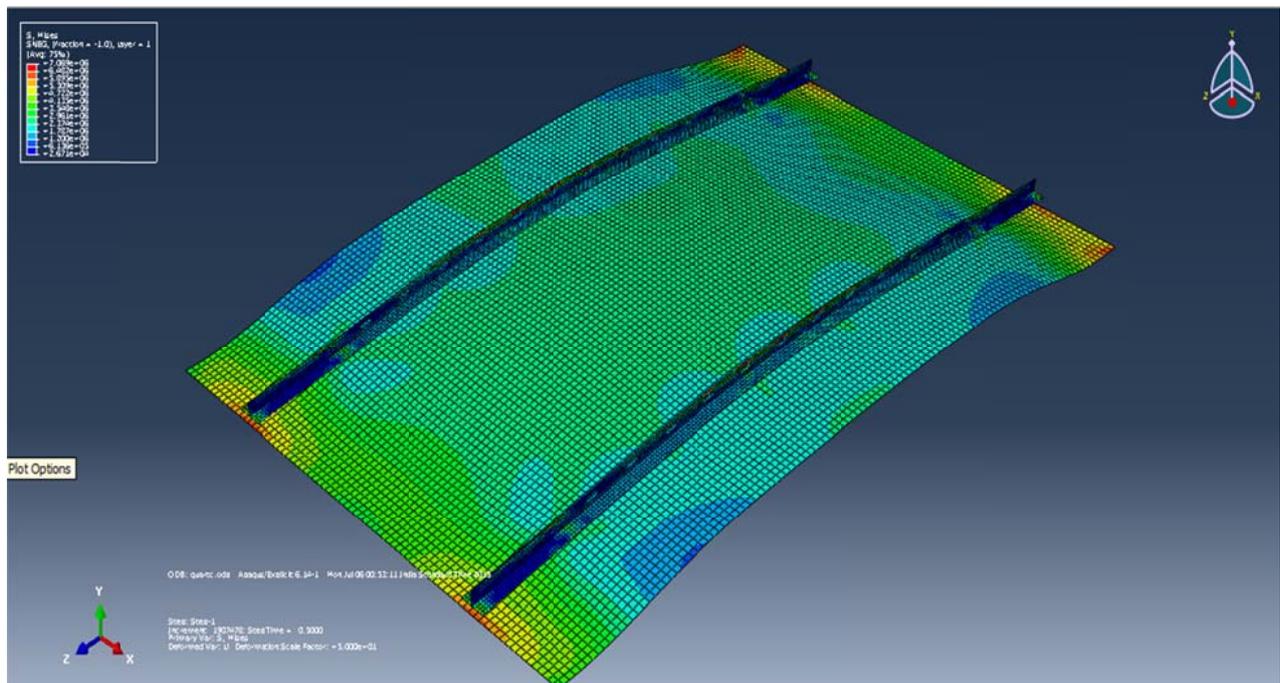


Fig. 11a Deformed Castellated plate for $(2343/1.414)x(\cos \mu t + \sin \mu t)$ @ $T=0.5s$

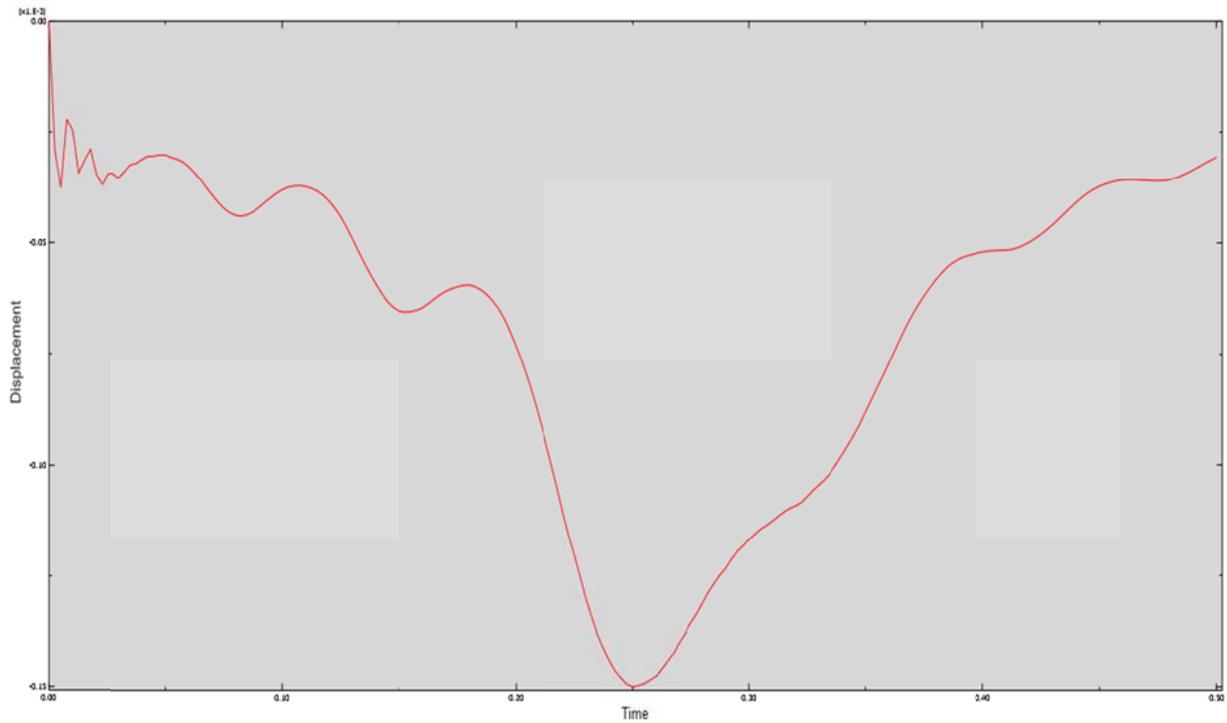


Fig 11b: Axial displacement behaviour of Castellated plate for $(2343/1.414) \times (\cos \mu t + \sin \mu t)$ @ T=0.5s

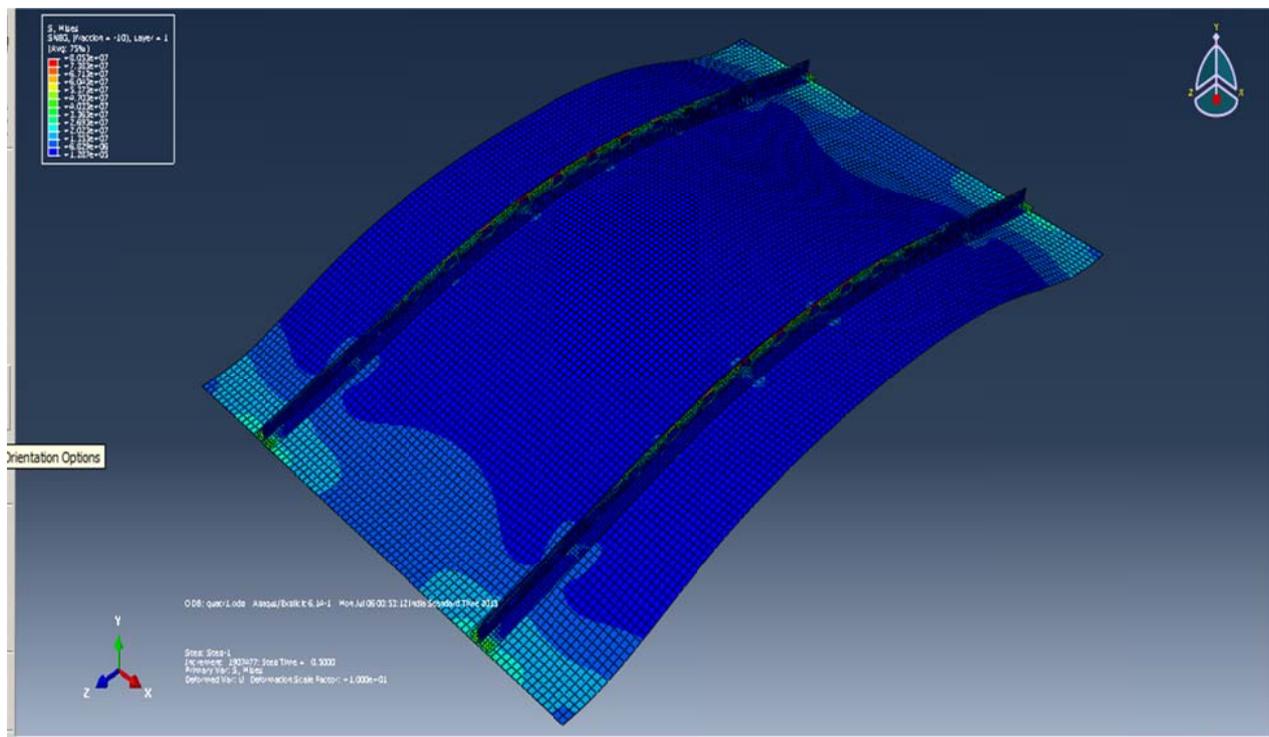


Fig. 12a Deformed Castellated plate for $2306(0.5+0.5\sin \mu t)$ @ T=0.5s

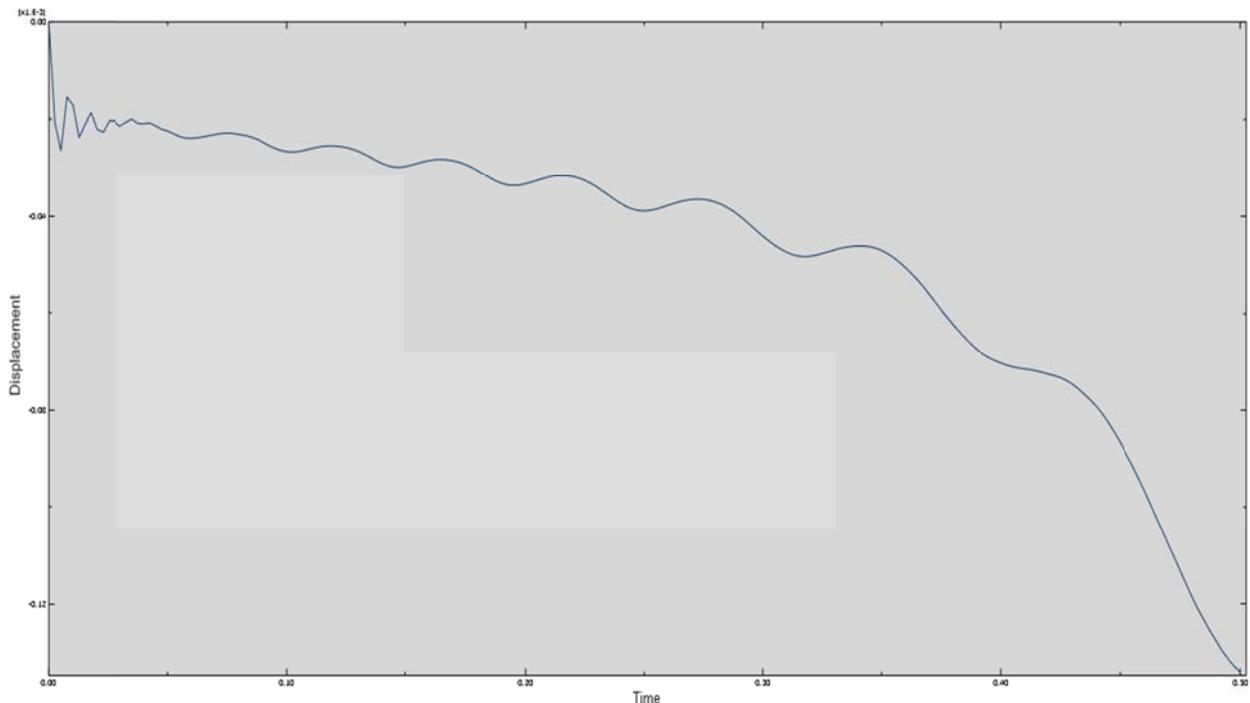


Fig 12b: Axial displacement behaviour of Castellated plate for $2306(0.5+0.5\sin\pi t)$ @ $T=0.5s$

4. CONCLUSION

The conclusions of the present study can be summarized as,

1. The dynamic buckling behaviour of laminated composite stiffened plate is studied numerically in the present investigation.
2. The sinusoidal functions with some initial amplitude have significant effect on dynamic buckling response.
3. The duration of the loading function have also significant influence on dynamic buckling response.
4. Present investigation is purely numerical. However, some experiments are needed for deeper and clear understanding.
5. Effect of various other parameters, like, lamination scheme, boundary conditions, effect of depth of stiffener, types of stiffener etc. can also be studied in the present investigation as a future scope.

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