





of variables. Inclusion of the final term is to ensure an exact approximation. But typically the expansion up to the second order will serve the purpose of approximate expansion by the assumption that contribution of higher order terms to the overall performance is negligible.

Consider the functional,

$$\int_{\Gamma} \left[ g(x) - g_0 + \sum_{i=1}^n g_i(x_i) + \sum_{1 \leq i_1 \leq i_2 \leq N} g_{i_1 i_2}(x_{i_1}, x_{i_2}) + \dots + \sum_{1 \leq i_1 < \dots < i_l \leq N} g_{i_1 \dots i_l}(x_{i_1}, x_{i_2}, \dots, x_{i_l}) + \dots + g_{12 \dots N}(x_1, x_2, \dots, x_n) \right]^2 d\mu(x)$$

Subjected to the constraint

$$\int_{[0,1]} g_{i_1 \dots i_l}(x_{i_1}, x_{i_2}, \dots, x_{i_l}) dx_{i_k} = 0$$

This functional is minimized to obtain the terms  $g_{i_1 i_2}(x_{i_1}, x_{i_2}, \dots, x_{i_l})$  uniquely. Major and popular HDMR techniques are ANOVA (ANalysis Of VAriance) HDMR, RS (Random Sampling)-HDMR, Cut-HDMR and ANOVA-HDMR based on Cut-HDMR expansions. In this paper, we are using cut-HDMR technique for approximating the nonlinear relations.

Cut-HDMR expands the multi-dimensional function  $g(x)$  with respect to a reference point ' $c$ ', chosen within the neighborhood of interest of the input variables and the final outcome is independent of ' $c$ ' at the convergence limit.

Cut HDMR yields:

$$g_0 = g(c) \quad (2a)$$

$$g_i(x_i) = g(x_i, c^i) - g_0 \quad (2b)$$

$$g_{i_1 i_2} = g(x_{i_1}, x_{i_2}, c^{i_1 i_2}) - g_{i_1}(x_{i_1}) - g_{i_2}(x_{i_2}) - g_0 \quad (2c)$$

And so on.

$g_0$  is evaluated at the reference point ' $c$ '.  $g(x_i, c^i) = g(c_1, c_2, \dots, c_{i-1}, x_i, c_{i+1}, \dots, c_N)$  denotes the function in terms of the input variable  $x_i$  alone, i.e. univariate and at all the other variables at their reference point value. The second order bivariate term  $g_{i_1 i_2}(x_{i_1}, x_{i_2})$  takes into account the cooperative effects of binary sets of input variables. Subtracting off the lower order terms from higher order terms is to ensure the unique contribution from each term.

First and second order approximations of  $g(x)$ , excluding the higher order residual terms and with reference to the reference point are as follows:

$$\tilde{g}(x) = g(x_1, x_2, \dots, x_N) = \sum_{i=1}^N g(c_1, c_2, \dots, c_{i-1}, x_i, c_{i+1}, \dots, c_N) - (N-1)g(c) \quad (3)$$

And

$$\begin{aligned}\tilde{g}(x) &= S_2 - (N - 2) \times S_1 + \frac{(N - 1)(N - 2)}{2} g(c) \\ S_2 &= \sum_{\substack{i_1=1, i_2=1 \\ i_1 < i_2}}^N g(c_1, c_2, \dots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \dots, \\ &\quad c_{i_2-1}, x_{i_2}, c_{i_2+1}, \dots, c_N) \\ S_1 &= \sum_{i=1}^N g(c_1, c_2, \dots, c_{i-1}, x_i, c_{i+1}, \dots, c_N)\end{aligned}\quad (4)$$

Along the cut points, this approximation gives exact values. For any arbitrary point in the domain, response surface models are required.

The first order interaction term for an arbitrary point  $x_i^{new}$  may be written down as:

$$g_i(x_i^{new}) = \sum_{j=1}^n \Phi_j(x_i^{new}) g(c_1, \dots, c_{i-1}, x_i^j, c_{i+1}, \dots, c_n) \quad (5)$$

Where  $\Phi_j(x_i^{new})$  is one dimensional Lagrange interpolation polynomial. Thus, the one dimensional response surface is generated with  $n=3,5,7, or 9 sample points that are uniformly distributed as  $\mu_i - (n - 1)\sigma_i/2, \mu_i - (n - 3)\sigma_i/2, \dots, \mu_i, \dots, \mu_i - (n + 1)\sigma_i/2, \mu_i - (n + 3)\sigma_i/2$ , along the variable axis  $x_i$  with the mean  $\mu_i$  and standard deviation  $\sigma_i$ .$

In a similar fashion, the second order interaction term for any two arbitrary points, is as follows:

$$\begin{aligned}g_{ij}(x_{i_1}^{new}, x_{i_2}^{new}) \\ &= \sum_{j=1}^n \Phi_{j_1 j_2}(x_{i_1}^{new}, x_{i_2}^{new}) \\ &\quad * g(c_1, \dots, c_{i_1-1}, x_{i_1}^{j_1}, c_{i_1+1}, \dots, c_{i_2-1}, x_{i_2}^{j_2}, c_{i_2+1}, \dots, c_n)\end{aligned}\quad (6)$$

Where  $\Phi_{j_1 j_2}(x_{i_1}^{new}, x_{i_2}^{new})$  is the tensor product of  $\Phi_{j_1}(x_{i_1}^{new})$  and  $\Phi_{j_2}(x_{i_2}^{new})$ .

This one is on a grid formed by taking  $n (= 3,5,7,$  or 9) sample points that are uniformly distributed as  $\mu_{i_1} - (n - 1)\sigma_{i_1}/2, \mu_{i_1} - (n - 3)\sigma_{i_1}/2, \dots, \mu_{i_1}, \dots, \mu_{i_1} + (n - 1)\sigma_{i_1}/2, \mu_{i_1} + (n - 3)\sigma_{i_1}/2$ , along the variable axis  $x_{i_1}$  with the mean  $\mu_{i_1}$  and standard deviation  $\sigma_{i_1}$ , and  $n = (3,5,7,$  or 9) sample points that are uniformly distributed as  $\mu_{i_2} - (n - 1)\sigma_{i_2}/2, \mu_{i_2} - (n - 3)\sigma_{i_2}/2, \dots, \mu_{i_2}, \dots, \mu_{i_2} + (n - 1)\sigma_{i_2}/2,$

$\mu_{i_2} + (n + 3)\sigma_{i_2}/2$  , along the variable axis  $x_{i_2}$  with the mean  $\mu_{i_2}$  and standard deviation  $\sigma_{i_2}$ .

In the proposed method, all the multidimensional integrals involved in Bayesian Inference problems are computed with the help of HDMR. For the same, the bounds of the parameters or variables of interest need to be specified. Thereof the integrands are approximated by HDMR expansions and then the integrals are evaluated.

## 2.2. Dynamic State estimation

Consider the dynamic system with the following nonlinear state and measurement model

$$x_{k+1} = f(x_k, u_k, w_k) \quad (7a)$$

$$z_{k+1} = h(x_{k+1}, v_{k+1}) \quad (7b)$$

Where  $x_k$  ,  $u_k$  and  $z_k$  are the state, input and measurement vectors.  $w_k$  and  $v_k$  are process noise and measurement noise respectively, which are assumed to be statistically independent of each other.

Vector of noisy measurements is represented as:

$$Z_{k+1} = \{z_1, z_2, \dots, z_k\} \quad (8)$$

From our prior knowledge of the system, we form the *a priori* PDF ,  $p(x_k|Z_k)$  of the states to be estimated. State transition probability is computed from the state model, incorporating Markovian process assumption. The Chapman-Kolmogorov equation is used in the prediction step as follows:

$$p(x_{k+1}|Z_k) = \int p(x_{k+1}|x_k) p(x_k|Z_k) dx_k \quad (9)$$

Update step computes the posterior density by assimilating the measurement data with the prior density through Bayes' theorem.

$$p(x_{k+1}|Z_{k+1}) = \frac{p(z_{k+1}|x_{k+1}) p(x_{k+1}|Z_k)}{p(z_{k+1}|Z_k)} \quad (10)$$

The normalizing constant which is the denominator term, is computed as follows:

$$p(z_{k+1}|Z_k) = \int p(z_{k+1}|x_{k+1}) p(x_{k+1}|Z_k) dx_{k+1} \quad (11)$$

The likelihood function,  $p(z_{k+1}|x_{k+1})$  is obtained from the measurement model , using basic concepts of probability.

Measurement noise may be uncertain and need to be modelled with respect to the available data. We characterize the measurement a priori as follows:

$$\hat{z}_{k+1|k} = \int h(x_{k+1}, u_{k+1})p(z_{k+1}|Z_k) dx_k \quad (12)$$

Represents the expected value of measurement.

Covariance is computed as:

$$P_{zz,k+1|k} = \int (h(x_{k+1}, u_{k+1}) - \hat{z}_{k+1|k})(h(x_{k+1}, u_{k+1}) - \hat{z}_{k+1|k})^T p(z_{k+1}|Z_k) dx_k + R_{k+1} \quad (13)$$

Where  $R_{k+1}$  is the assumed measurement noise.

In the proposed method, all the multidimensional integrals involved in Bayesian Inference problems are computed with the help of HDMR. For the same, the bounds of the parameters or variables of interest need to be specified. Thereof the integrands are approximated by HDMR expansions and then the integrals are evaluated.

### 3. NUMERICAL EXAMPLE

(Au & Beck,2002)

Here, we are considering the identification of inter story stiffness values of a two story structural shear frame as shown in figure. The frame is with a bay width of 5.0 m and story height of 2.5 m. The Young's modulus is taken to be 200 GPa and mass density as 7800 kg/m<sup>3</sup>. The beams have a cross-sectional area of 0.01 m<sup>2</sup> and a moment of inertia of 6.0 × 10<sup>-4</sup> m<sup>4</sup>. For columns, cross-sectional area and mass density are 0.02 m<sup>2</sup> and 1.5 × 10<sup>-4</sup> m<sup>4</sup> respectively. The structure has modal frequencies of 5.20 and 15.4 Hz. The structure is assumed to have 1.0% of critical damping for all modes. In the two degree of freedom model being used for identification, the stiffness matrix is given by:

$$K = K_\eta \begin{bmatrix} \theta_1 + \theta_2 & -\theta_2 \\ -\theta_2 & \theta_2 \end{bmatrix}$$

where  $\theta_i$  is the inter-story stiffness parameter of the  $i^{th}$  story . Identification of these inter story stiffness parameters need to be done , with  $K_\eta = 46.08 \text{ MN/m}$ .

A lumped mass model (i.e., diagonal mass matrix) with a floor mass of 11.17 × 10<sup>3</sup> kg, is considered. There is no combination of ( $\theta_1, \theta_2$ ) such that the structural mass and stiffness matrix are fully consistent with the actual frame. The nominal and other

possible models deviate from the actual frame system and this is imposed deliberately to simulate the reality.

Modal frequencies are taken as the observation data D.

$$\omega^{(1)} = 5.5 \text{ Hz} ; \quad \omega^{(2)} = 14.9 \text{ Hz}$$

Identification of inter story stiffness parameters by the proposed method is compared against Markov Chain Monte Carlo method solutions and Unscented Kalman Filter solutions.

### 3.1. By Markov Chain Monte Carlo Method

The prior PDF is taken to be independent log-normal PDFs with means of 0.9 and 1.2 and unit variance. Using the modal data D, the updated PDF or posterior PDF for the stiffness parameter vector  $\theta$  is formulated as:

$$p_D = k_0 \exp \left[ -\frac{J(\theta)}{\phi^2} \right] p_\theta$$

where  $k_0$  is a normalizing constant,  $\phi$  is related to the prediction-error coefficient of variation and the modal goodness-of-fit function depends on the type of measurements.  $p_\theta$  is the prior PDF for the inter story stiffness parameters. Here, these parameters are assumed to be distributed log-normally with mean of 0.9 and 1.2 and variance unity.

In the adaptive procedure, the sequence of intermediate PDFs  $\{p^{(s)}\}$  is constructed by successively substituting the sequence of values  $\phi^2 = 1/2s$  into the expression for  $p_D$ , for  $s = 1, 2, \dots, 10$ .

For the locally identifiable case, the goodness of fit function is

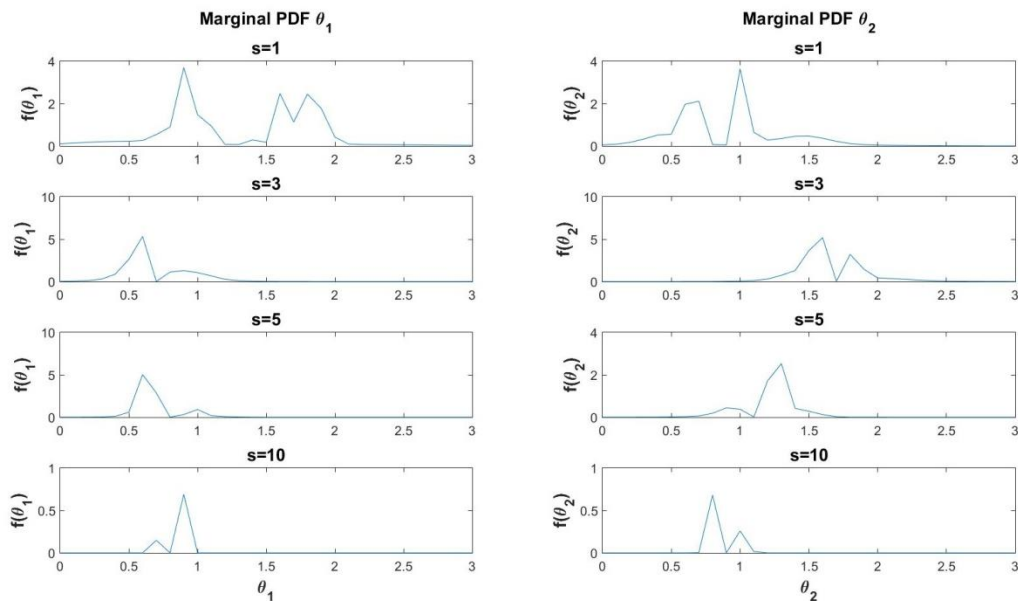
$$J(\theta) = \sum_{m=1}^2 \left[ \frac{\omega^{(m)2} - \omega^{(m)}(\theta)^2}{\omega^{(m)2}} \right]^2$$

where  $\omega^{(m)}$  and  $\omega^{(m)}(\theta)$  are the measured modal frequency and the model modal frequency of the  $m^{\text{th}}$  mode, which depend on the structural parameters in  $\theta$ .

Following the adaptive MCMC method, Markov chain samples are simulated for each of the parameters and then the kernel densities are constructed. Marginal PDFs of each of the variables are computed to get the most probable value of the parameters need to be identified.

It can be observed that the values are tending to be in a narrower range, towards the final target PDF.

Marginal densities which are constructed without integration, from the samples directly is as shown in figure below. The peaks represent the most probable values of the parameters. Hence the identified values  $(\theta_1, \theta_2)$  here are (0.7,0.8) and (0.9,1.0).



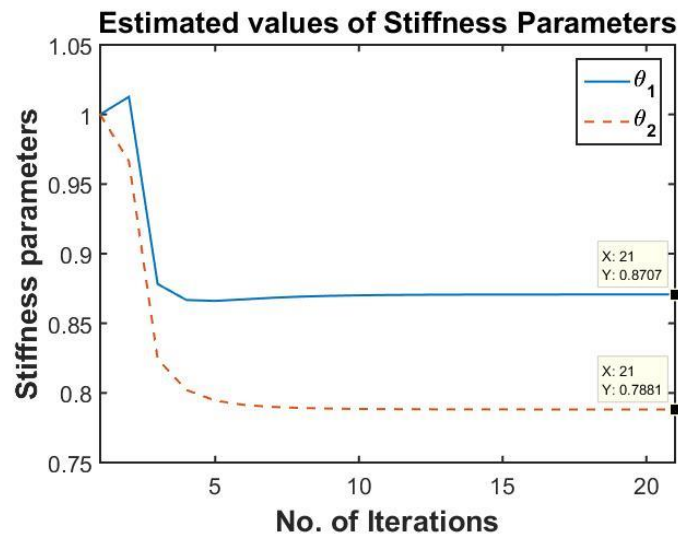
**Fig. 1** Marginal densities of  $\theta_1$  and  $\theta_2$ , peaked at 0.9 and 0.8 respectively

### 3.2. By the proposed HDMR-Bayesian Technique

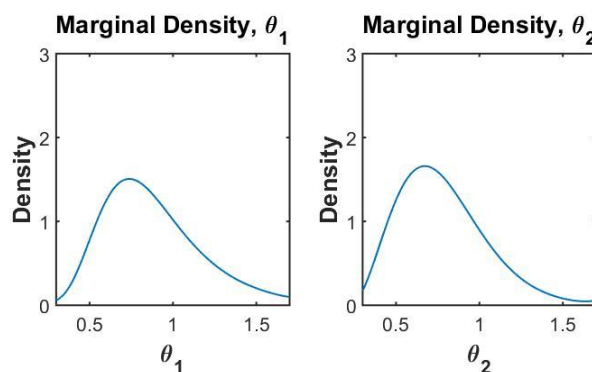
Next, we solved the system with the proposed method. Here, we used Log-normal prior density. In this method, we specify certain bounds for the uncertain quantities, based on our prior knowledge on the system. The densities when updated to posterior densities are truncated to these bounds, with a computed normalizing constant. It is not clear how well the final posterior density represents the true nature of the system. We could say that the results are well valid within the bounds.

The results obtained are comparable with the MCMC results. Unlike the sampling technique, this method gives you a unique point estimate with a bound specified. In either case, the estimates of the respective stiffness parameters tend to the values with higher probability as obtained in MCMC method. The proposed method shows a faster or stable convergence with respect to the other methods used. HDMR expansion used in this example is of 2<sup>nd</sup> order with 9 sample points.





**Fig. 2** Estimated mean values of  $\theta_1$  and  $\theta_2$  by Bayesian HDMR technique, with log-normal prior density



**Fig. 3** Marginal densities function of  $\theta_1$  and  $\theta_2$  by Bayesian HDMR technique, with Log-normal prior density

#### 4. CONCLUSIONS & FUTURE WORK

We have many large scale applications where the computational effort is much higher with respect to the current methods, though sampling based DAs are a significant leap. Non Gaussian error modeling is another challenge where Gaussian filters cannot be used. The present method is expected to be a better, faster and accurate performer, in the multi-dimensional and non-Gaussian models, in addition to its possibilities on non-linearity. These aspects need to be investigated ,explored and studied. This will be particularly useful for real time applications with sequentially arriving measurement data where online state/model updating needs faster algorithms.

The proposed method converges at a faster rate and algorithm is faster, simpler and more direct. This results in significant reduction in computational cost. This makes the proposed method more advantageous for its use in large scale problems.

## REFERENCES

- Beck, J.L., Au, S.K. (2002), "Bayesian Updating of Structural Models and Reliability using Markov Chain Monte Carlo Simulation ", *J. Eng. Mech., ASCE*, 128(4), 380-391.
- Beck, J.L., Katafygiotis, L.S. (1998), "Updating models and their uncertainties I: Bayesian statistical framework", *J. Eng. Mech., ASCE*, 124(4), 455-461.
- Beck, J.L., Au, S.K. (1999), "A new adaptive importance sampling scheme for reliability calculations", *Structural Safety*, 21 ,135-158.
- Chowdhury, R., Rao, B. N., Prasad, A. M. (2009), "High-dimensional model representation for structural reliability analysis", *Commun. Numer. Meth. Engng, Wiley*, 25, 301-337.
- Grewal,M.S., Andrews, A.P., "Kalman Filtering: Theory and Practice Using MATLAB", *Wiley*.
- Rabitz, H., Alis, O.F. (1999), "General foundations of high-dimensional model representations", *Journal of Mathematical Chemistry*, 25, 197–233.
- Yuen,KV. (2010), "Bayesian methods for structural dynamics and civil engineering", *Wiley*.