of variables. Inclusion of the final term is to ensure an exact approximation. But typically the expansion up to the second order will serve the purpose of approximate expansion by the assumption that contribution of higher order terms to the overall performance is negligible.

Consider the functional,

$$\int_{\Gamma}\left[g(x)-g_0+\sum_{i=1}^{n}g_i(x_i)+\sum_{1\le i_1\le i_2\le N}g_{i_1i_2}\left(x_{i_1},x_{i_2}\right)+\ldots+\sum_{1\le i_1<\cdots<i_l\le N}g_{i_1\ldots i_l}\left(x_{i_1},x_{i_2},\ldots,x_{i_l}\right)+\cdots+g_{12\ldots N}(x_1,x_2,\ldots,x_n)\right]^2 d\mu(x)$$

Subjected to the constraint

$$\int_{[0,1]}g_{i_1\ldots i_l}\left(x_{i_1},x_{i_2},\ldots,x_{i_l}\right)dx_{i_k}=0$$

This functional is minimized to obtain the terms $g_{i_1i_2}\left(x_{i_1},x_{i_2},\ldots,x_{i_l}\right)$ uniquely. Major and popular HDMR techniques are ANOVA (ANalysis Of VAriance) HDMR, RS (Random Sampling)-HDMR, Cut-HDMR and ANOVA-HDMR based on Cut-HDMR expansions. In this paper, we are using cut-HDMR technique for approximating the nonlinear relations.

Cut-HDMR expands the multi-dimensional function g(x) with respect to a reference point '*c*', chosen within the neighborhood of interest of the input variables and the final outcome is independent of '*c*' at the convergence limit.

Cut HDMR yields:

$$g_0=g(c) \tag{2a}$$

$$g_i(x_i)=g\left(x_i,c^i\right)-g_0 \tag{2b}$$

$$g_{i_1i_2}=g\left(x_{i_1},x_{i_2},c^{i_1i_2}\right)-g_{i_1}\left(x_{i_1}\right)-g_{i_2}\left(x_{i_2}\right)-g_0 \tag{2c}$$

And so on.

g_0 is evaluated at the reference point 'c'. $g\left(x_i,c^i\right)=g(c_1,c_2,\ldots,c_{i-1},x_i,c_{i+1},\ldots,c_N)$ denotes the function in terms of the input variable x_i alone, i.e. univariate and at all the other variables at their reference point value. The second order bivariate term $g_{i_1i_2}\left(x_{i_1},x_{i_2}\right)$ takes into account the co-operative effects of binary sets of input variables. Subtracting off the lower order terms from higher order terms is to ensure the unique contribution from each term.

First and second order approximations of g(x), excluding the higher order residual terms and with reference to the reference point are as follows:

$$\tilde{g}(x)=g(x_1,x_2,\ldots,x_N)=\sum_{i=1}^{N}g(c_1,c_2,\ldots,c_{i-1},x_i,c_{i+1},\ldots,c_N)\ -(N-1)g(c) \tag{3}$$

And

$$\tilde{g}(x) = S_2 - (N-2) \times S_1 + \frac{(N-1)(N-2)}{2} g(c)$$
$$S_2 = \sum_{\substack{i_1=1, i_2=1 \\ i_1<i_2}}^{N} g\begin{pmatrix} c_1, c_2, \dots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \dots \\ , c_{i_2-1}, x_{i_2}, c_{i_2+1}, \dots, c_N \end{pmatrix} \quad (4)$$
$$S_1 = \sum_{i=1}^{N} g(c_1, c_2, \dots, c_{i-1}, x_i, c_{i+1}, \dots, c_N)$$

Along the cut points, this approximation gives exact values. For any arbitrary point in the domain, response surface models are required.

The first order interaction term for an arbitrary point x_i^{new} may be written down as:

$$g_i(x_i^{new}) = \sum_{j=1}^{n} \Phi_j(x_i^{new})\, g(c_1, \dots, c_{i-1}, x_i^j, c_{i+1}, \dots, c_n) \quad (5)$$

Where $\Phi_j(x_i^{new})$ is one dimensional Lagrange interpolation polynomial. Thus, the one dimensional response surface is generated with n=3,5,7,or 9 sample points that are uniformly distributed as $\mu_i - (n-1)\sigma_i/2,\ \mu_i - (n-3)\sigma_i/2, \dots, \mu_i, \dots, \mu_i - (n+1)\sigma_i/2,\ \mu_i - (n+3)\sigma_i/2$, along the variable axis x_i with the mean μ_i and standard deviation σ_i.

In a similar fashion, the second order interaction term for any two arbitrary points, is as follows:

$$g_{ij}\left(x_{i_1}^{new}, x_{i_2}^{new}\right) = \sum_{j=1}^{n} \Phi_{j_1 j_2}\left(x_{i_1}^{new}, x_{i_2}^{new}\right) * g(c_1, \dots, c_{i_1-1}, x_{i_1}^{j_1}, c_{i_1+1}, \dots, c_{i_2-1}, x_{i_2}^{j_2}, c_{i_2+1}, \dots, c_n) \quad (6)$$

Where $\Phi_{j_1 j_2}\left(x_{i_1}^{new}, x_{i_2}^{new}\right)$ is the tensor product of $\Phi_{j_1}\left(x_{i_1}^{new}\right)$ and $\Phi_{j_2}\left(x_{i_2}^{new}\right)$.

This one is on a grid formed by taking n (= 3,5,7,or 9) sample points that are uniformly distributed as $\mu_{i_1} - (n-1)\sigma_{i_1}/2,\ \mu_{i_1} - (n-3)\sigma_{i_1}/2, \dots, \mu_{i_1}, \dots, \mu_{i_1} + (n-1)\sigma_{i_1}/2,\ \mu_{i_1} + (n-3)\sigma_{i_1}/2$, along the variable axis x_{i_1} with the mean μ_{i_1} and standard deviation σ_{i_1}, and n = (3,5,7,or 9) sample points that are uniformly distributed as $\mu_{i_2} - (n-1)\sigma_{i_2}/2,\ \mu_{i_2} - (n-3)\sigma_{i_2}/2, \dots, \mu_{i_1}, \dots, \mu_{i_2} + (n-1)\sigma_{i_2}/2,$

$\mu_{i_2} + (n+3)\sigma_{i_2}/2$, along the variable axis x_{i_2} with the mean μ_{i_2} and standard deviation σ_{i_2}.

In the proposed method, all the multidimensional integrals involved in Bayesian Inference problems are computed with the help of HDMR. For the same, the bounds of the parameters or variables of interest need to be specified. Thereof the integrands are approximated by HDMR expansions and then the integrals are evaluated.

2.2. Dynamic State estimation

Consider the dynamic system with the following nonlinear state and measurement model

$$x_{k+1} = f(x_k, u_k, w_k) \tag{7a}$$

$$z_{k+1} = h(x_{k+1}, v_{k+1}) \tag{7b}$$

Where x_k , u_k and z_k are the state, input and measurement vectors. w_k and v_k are process noise and measurement noise respectively, which are assumed to be statistically independent of each other.

Vector of noisy measurements is represented as:

$$Z_{k+1} = \{z_1, z_2, \dots, z_k\} \tag{8}$$

From our prior knowledge of the system, we form the *a priori* PDF , $p(x_k|Z_k)$ of the states to be estimated. State transition probability is computed from the state model, incorporating Markovian process assumption. The Chapman-Kolmogorov equation is used in the prediction step as follows:

$$p(x_{k+1}|Z_k) = \int p(x_{k+1}|x_k)\, p(x_k|Z_k)\, dx_k \tag{9}$$

Update step computes the posterior density by assimilating the measurement data with the prior density through Bayes' theorem.

$$p(x_{k+1}|Z_{k+1}) = \frac{p(z_{k+1}|x_{k+1})\, p(x_{k+1}|Z_k)}{p(z_{k+1}|Z_k)} \tag{10}$$

The normalizing constant which is the denominator term, is computed as follows:

$$p(z_{k+1}|Z_k) = \int p(z_{k+1}|x_{k+1}) p(x_{k+1}|Z_k)\, dx_{k+1} \tag{11}$$

The likelihood function, $p(z_{k+1}|x_{k+1})$ is obtained from the measurement model , using basic concepts of probability.

Measurement noise may be uncertain and need to be modelled with respect to the available data. We characterize the measurement a priori as follows:

$$\hat{z}_{k+1|k} = \int h(x_{k+1}, u_{k+1}) p(z_{k+1}|Z_k)\, dx_k \tag{12}$$

Represents the expected value of measurement.

Covariance is computed as:

$$P_{zz,k+1|k} = \int (h(x_{k+1}, u_{k+1}) - \hat{z}_{k+1|k})(h(x_{k+1}, u_{k+1}) - \hat{z}_{k+1|k})^T p(z_{k+1}|Z_k)\, dx_k + R_{k+1} \tag{13}$$

Where R_{k+1} is the assumed measurement noise.

In the proposed method, all the multidimensional integrals involved in Bayesian Inference problems are computed with the help of HDMR. For the same, the bounds of the parameters or variables of interest need to be specified. Thereof the integrands are approximated by HDMR expansions and then the integrals are evaluated.

3. NUMERICAL EXAMPLE

(Au & Beck,2002)

Here, we are considering the identification of inter story stiffness values of a two story structural shear frame as shown in figure. The frame is with a bay width of 5.0 m and story height of 2.5 m. The Young's modulus is taken to be 200 GPa and mass density as 7800 kg/m^3. The beams have a cross-sectional area of 0.01 m^2 and a moment of inertia of 6.0 × 10^{-4} m^4 . For columns, cross-sectional area and mass density are 0.02 m^2 and 1.5 × 10^{-4} m^4 respectively. The structure has modal frequencies of 5.20 and 15.4 Hz. The structure is assumed to have 1.0% of critical damping for all modes. In the two degree of freedom model being used for identification, the stiffness matrix is given by:

$$K = K_\eta \begin{bmatrix} \theta_1 + \theta_2 & -\theta_2 \\ -\theta_2 & \theta_2 \end{bmatrix}$$

where θ_i is the inter-story stiffness parameter of the i^{th} story . Identification of these inter story stiffness parameters need to be done , with $K_\eta = 46.08\, MN/m$.

A lumped mass model (i.e., diagonal mass matrix) with a floor mass of 11.17 × 10^3 kg, is considered. There is no combination of (θ_1, θ_2) such that the structural mass and stiffness matrix are fully consistent with the actual frame. The nominal and other

possible models deviate from the actual frame system and this is imposed deliberately to simulate the reality.

Modal frequencies are taken as the observation data D.

$\omega^{(1)} = 5.5\,Hz$; $\omega^{(2)} = 14.9\,Hz$

Identification of inter story stiffness parameters by the proposed method is compared against Markov Chain Monte Carlo method solutions and Unscented Kalman Filter solutions.

3.1. By Markov Chain Monte Carlo Method

The prior PDF is taken to be independent log-normal PDFs with means of 0.9 and 1.2 and unit variance. Using the modal data D, the updated PDF or posterior PDF for the stiffness parameter vector θ is formulated as:

$$p_D = k_0 exp\left[-\frac{J(\theta)}{\varphi^2}\right] p_\theta$$

where k_0 is a normalizing constant, ϕ is related to the prediction-error coefficient of variation and the modal goodness-of-fit function depends on the type of measurements. p_θ is the prior PDF for the inter story stiffness parameters . Here, these parameters are assumed to be distributed log-normally with mean of 0.9 and 1.2 and variance unity.

In the adaptive procedure, the sequence of intermediate PDFs $\{p^{(s)}\}$ is constructed by successively substituting the sequence of values ϕ^2 = 1/2s into the expression for p_D, for s = 1, 2, . . . , 10.

For the locally identifiable case, the goodness of fit function is

$$J(\theta) = \sum_{m=1}^{2}\left[\frac{\omega^{(m)^2} - \omega^{(m)}(\theta)^2}{\omega^{(m)^2}}\right]^2$$

where $\omega^{(m)}$ and $\omega^{(m)}(\theta)$ are the measured modal frequency and the model modal frequency of the m^{th} mode, which depend on the structural parameters in θ.

Following the adaptive MCMC method, Markov chain samples are simulated for each of the parameters and then the kernel densities are constructed. Marginal PDFs of each of the variables are computed to get the most probable value of the parameters need to be identified.

It can be observed that the values are tending to be in a narrower range, towards the final target PDF.

Marginal densities which are constructed without integration , from the samples directly is as shown in figure below. The peaks represent the most probable values of the parameters. Hence the identified values (θ_1, θ_2) here are (0.7,0.8) and (0.9,1.0).

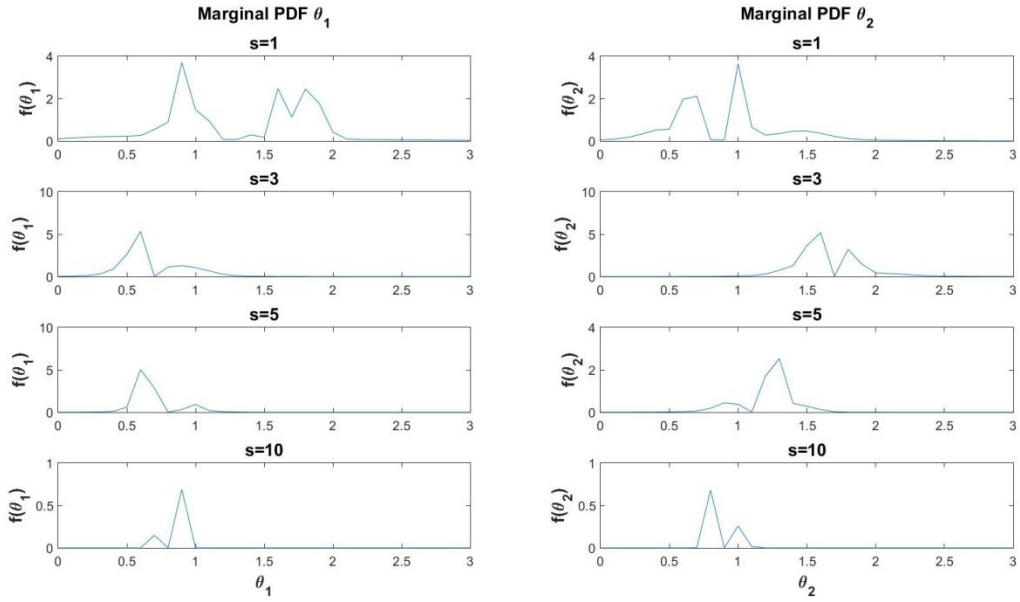

Fig. 1 Marginal densities of θ_1 and θ_2 , peaked at 0.9 and 0.8 respectively

3.2. By the proposed HDMR-Bayesian Technique

Next, we solved the system with the proposed method. Here, we used Log-normal prior density. In this method, we specify certain bounds for the uncertain quantities, based on our prior knowledge on the system. The densities when updated to posterior densities are truncated to these bounds, with a computed normalizing constant. It is not clear how well the final posterior density represents the true nature of the system. We could say that the results are well valid within the bounds.

The results obtained are comparable with the MCMC results. Unlike the sampling technique, this method gives you a unique point estimate with a bound specified. In either case, the estimates of the respective stiffness parameters tend to the values with higher probabilty as obtained in MCMC method. The proposed method shows a faster or stable convergence with respect to the other methods used. HDMR expansion used in this example is of 2^{nd} order with 9 sample points.

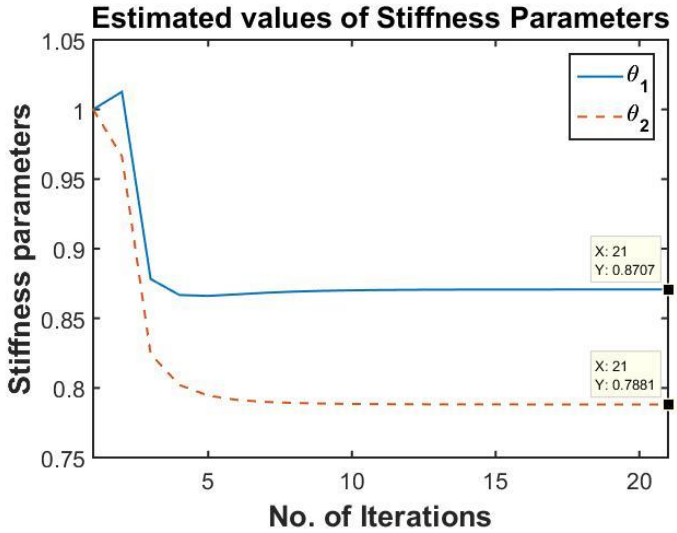

Fig. 2 Estimated mean values of θ_1 and θ_2 by Bayesian HDMR technique, with log-normal prior density

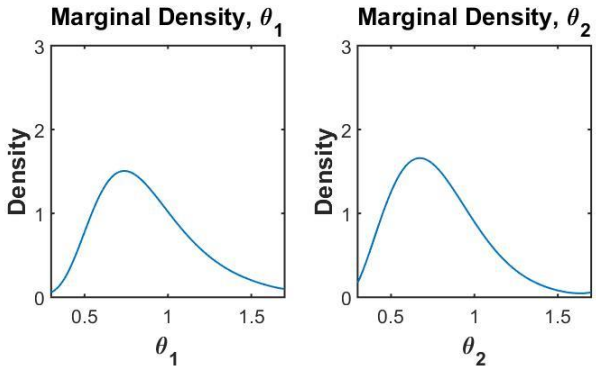

Fig. 3 Marginal densities function of θ_1 and θ_2 by Bayesian HDMR technique, with Log-normal prior density

4. CONCLUSIONS & FUTURE WORK

We have many large scale applications where the computational effort is much higher with respect to the current methods, though sampling based DAs are a significant leap. Non Gaussian error modeling is another challenge where Gaussian filters cannot be used. The present method is expected to be a better, faster and accurate performer, in the multi-dimensional and non-Gaussian models, in addition to its possibilities on non-linearity. These aspects need to be investigated ,explored and studied. This will be particularly useful for real time applications with sequentially arriving measurement data where online state/model updating needs faster algorithms.

The proposed method converges at a faster rate and algorithm is faster, simpler and more direct. This results in significant reduction in computational cost. This makes the proposed method more advantageous for its use in large scale problems.

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