









$$W(x-x_i, y-y_i) = \begin{cases} \frac{2}{3} - 4\bar{d}^2 + 4\bar{d}^3 & \text{if } \bar{d} \leq \frac{1}{2}, \\ \frac{4}{3} - 4\bar{d} + 4\bar{d}^2 - \frac{4}{3}\bar{d}^3 & \text{if } \frac{1}{2} < \bar{d} \leq 1, \\ 0 & \text{if } 1 < \bar{d}, \end{cases} \quad (8)$$

where  $(x_i, y_i)$  is the coordinates of node  $i$ ,  $\bar{d} = \sqrt{(x-x_i)^2 + (y-y_i)^2} / d_m$  and  $d_m$  is the size of the support.

Substituting Eq. (4) into Eqs. (3) and minimizing  $\Pi$  with respect to coefficients  $u_{\gamma ik}$  yield  $3 \times N_p \times N_z$  linear algebraic equations, which form a generalized eigenvalue problem with eigenvalue  $\omega^2$ .

#### 4. Convergence study

The accuracy and efficiency of the presented approach are demonstrated through convergence study and comparisons of the present results with those published data. In the following analyses, the nodes for constructing  $\phi_{\gamma i}(x, y)$  are uniformly distributed inside the domain,  $\tilde{x} \in [0.005a, 0.995a]$  and  $y \in [0.005b, 0.995b]$  with  $\Delta\tilde{x} = 0.99a/N_{d,a}$  and  $\Delta y = 0.99b/N_{d,b}$ , where  $N_{d,a}$  and  $N_{d,b}$  are parameters to be prescribed to define the distance between two adjacent nodes (see Fig. 2). When  $a=b$ , set  $N_{d,a} = N_{d,b} = N_d$ .

Table 2 shows the convergence of non-dimensional frequencies for a cantilevered FGM square plate with  $h/b=0.1$  made of aluminum (Al) and ceramic (alumina ( $Al_2O_3$ )) with a vertical central crack having crack length equal to  $0.5b$ . The parameter of volume fraction ( $\bar{m}$ ) is set 0.5. The non-dimensional frequency parameter  $\omega(b^2/h)\sqrt{\rho_c/E_c}$ , where subscript "c" denotes ceramic, is considered. The results were obtained using  $N_p = N_d^* N_d$  with  $N_d = 15, 20, 25, 30$  and  $40$ ,  $N_z = 3, 4$  and  $5$ , and the supports of the shape functions for in-plane and out-of-plane displacements equal to  $0.3b$  and  $0.8b$ , respectively. Table 2 illustrates that the numerical results converge from the upper bounds as  $N_d$  and  $N_z$  increase. The results obtained using  $N_d=30$  and  $N_z=5$  excellently agree with the results of Huang *et al.* (2012), who utilized the Ritz method with the admissible functions consisting of orthogonal polynomials and crack functions. Notably, since the admissible functions in Huang *et al.* (2012) include very high order of polynomials and trigonometric functions and cover the whole domain of plate, variables with 128-bit precision (with approximately 34 decimal digit accuracy) were used in their computer programs to avoid the numerical difficulties before the solutions converge.

## 5. Numerical Results

After the present solutions are validated, the proposed approach was utilized to investigate the vibrations of skewed cantilevered rhombic FGM plates having central vertical cracks with various lengths and different the volume fraction of the constituents. The following results were obtained using  $N_d=30$  and  $N_z=5$ .

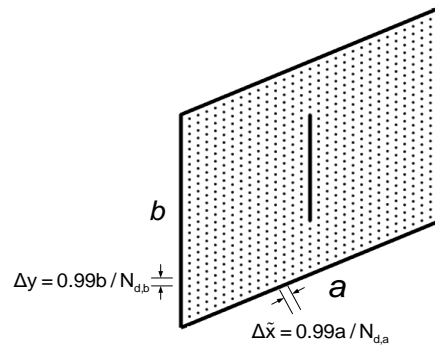


Fig. 2 Distribution of nodal points

Table 2 Convergence of frequency parameters  $\omega(b^2/h)\sqrt{\rho_c/E_c}$  for a cantilevered square FGM plate ( $h/b = 0.1$ ) having a vertical central crack

Mode	$N_d @ ( ) N_z = 3 ; [ ] N_z = 4 ; \{ \} N_z = 5$					Huang <i>et al.</i> (2012)
	15	20	25	30	40	
1	(0.6463) [0.6437] {0.6436}	(0.6456) [0.6430] {0.6429}	(0.6451) [0.6425] {0.6424}	(0.6450) [0.6425] {0.6423}	(0.6448) [0.6423] {0.6421}	0.6423
2	(1.582) [1.565] {1.564}	(1.579) [1.562] {1.562}	(1.579) [1.561] {1.561}	(1.578) [1.561] {1.561}	(1.578) [1.561] {1.560}	1.561
3	(3.427) [3.386] {3.385}	(3.418) [3.377] {3.376}	(3.410) [3.371] {3.369}	(3.409) [3.370] {3.368}	(3.406) [3.367] {3.365}	3.365
4	(3.965) [3.964] {3.964}	(3.962) [3.961] {3.961}	(3.961) [3.960] {3.959}	(3.961) [3.959] {3.959}	(3.960) [3.959] {3.958}	3.960
5	(4.856) [4.815] {4.814}	(4.853) [4.813] {4.812}	(4.851) [4.811] {4.810}	(4.851) [4.811] {4.809}	(4.851) [4.810] {4.809}	4.808

Table 3 Frequency parameters  $\omega(b^2/h)\sqrt{\rho_c/E_c}$  for skewed cantilevered rhombic FGM plates with vertical central cracks of various lengths ( $h/b = 0.1$ )

$\beta$	$\bar{m}$	$d/b$	Mode				
			1	2	3	4	5
$0^\circ$	0	0	1.041	2.444	6.102	6.598*	7.730
		0.1	1.038	2.439	6.053	6.579*	7.704
		0.3	1.019	2.419	5.726	6.422*	7.567
		0.5	0.9794 (0.9800)	2.397 (2.398)	5.189 (5.185)	6.093* (6.096*)	7.398 (7.398)
	0.2	0	0.9666	2.273	5.676	6.285*	7.190
		0.1	0.9641	2.268	5.631	6.268*	7.167
		0.3	0.9464	2.250	5.328	6.119*	7.039
		0.5	0.9098 (0.9100)	2.231 (2.231)	4.829 (4.826)	5.806* (5.828*)	6.882 (6.822)
	5	0	0.6838	1.594	3.966	4.285*	5.026
		0.1	0.6819	1.591	3.934	4.273*	5.009
		0.3	0.6690	1.576	3.718	4.172*	4.919
		0.5	0.6423 (0.6423)	1.561 (1.561)	3.368 (3.365)	3.959* (3.960*)	4.809 (4.809)
$15^\circ$	0	0	1.073	2.496	6.335	6.631*	7.477
		0.1	1.070	2.490	6.285	6.612*	7.464
		0.3	1.050	2.470	5.951	6.445*	7.390
		0.5	1.008	2.446	5.392	6.091*	7.262
	0.2	0	0.9964	2.321	5.895	6.318*	6.957
		0.1	0.9937	2.316	5.849	6.300*	6.945
		0.3	0.9749	2.298	5.539	6.141*	6.875
		0.5	0.9366	2.277	5.021	5.805*	6.757
	5	0	0.7047	1.628	4.108	4.314*	4.860
		0.1	0.7027	1.624	4.076	4.301*	4.852
		0.3	0.6888	1.610	3.858	4.190*	4.804
		0.5	0.6609	1.593	3.495	3.961*	4.721

Table 3 lists the first five non-dimensional frequency parameters of skewed cantilevered rhombic FGM plates with  $h/b=0.1$  and skew angle  $\beta = 0^\circ$  and  $15^\circ$  and having vertical central cracks with crack lengths  $d/b=0, 0.1, 0.3$  and  $0.5$ , while Table 4 shows the results for plates with  $\beta = 30^\circ$  and  $h/b=0.1$  and  $0.2$ . In both tables, the volume fraction  $\bar{m}=0, 0.2$  and  $5$  are under consideration. Notably, “\*” denotes in-plane

deformation dominated modes.

In Table 3, the parenthesized results were obtained by Huang *et al.* (2012) using three-dimensional elasticity theory and the conventional Ritz method, while the parenthesized results in Table 4 were obtained by McGee and Butalia (1994) utilizing higher-order shear deformable plate theory and the finite element approach. Comparisons of the present results with those published results reveal the correctness and accuracy of the present results.

Tables 3 and 4 show that, as expected, the non-dimensional frequency parameters decrease with the increase of crack length. A crack with  $d/b=0.1$  only reduce the first five frequencies by less than 1%, while a  $d/b=0.5$  crack can reduce frequencies by 15%. The frequencies decrease with the increase of the volume fraction  $\bar{m}$  because increasing  $\bar{m}$  reduces the stiffness more than it does the mass of plate. The first four frequencies increase as the skew angle increases, except for some results for  $d/b=0.5$ . Table 4 displays the dimensionless frequency parameters decrease with the increase of thickness  $h$  because  $h$  is involved in the definition of the non-dimensional frequency parameter.

## 6. CONCLUSIONS

Proposed herein is a three-dimensional elasticity-based Ritz procedure to obtain accurate vibration frequencies of skewed cantilevered rhombic FGM plates with vertical internal cracks. The admissible functions are constructed by MLS with proposed sets of enriched basis functions, which consists of regular polynomials and crack functions that properly describe the 3-D stress singularities at the terminus edge fronts of the crack and show displacement discontinuities across the crack.

The efficiency of the proposed solutions has been substantiated through convergence study and comparisons with the published results. The present 3-D approach has been employed to investigate the effects of volume fraction ( $\bar{m}$ ), crack length ratios ( $d/b$ ), thickness-to-length ratios ( $h/b$ ) and skew angles ( $\beta$ ) on the vibration frequencies of skewed cantilevered rhombic FGM plates with vertical central cracks. Most of the results are first shown in the literature and can be used as standard to judge the accuracy of other numerical methods and various plate theories.

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Table 4 Frequency parameters  $\omega(b^2/h)\sqrt{\rho_c/E_c}$  for skewed cantilevered rhombic FGM plates with  $\beta=30^\circ$  and having vertical central cracks

$h/b$	$\bar{m}$	$d/b$	Mode				
			1	2	3	4	5
0.1	0	0	1.171 (1.168)	2.692 (2.686)	6.684* (6.692*)	7.063 (7.047)	7.364 (7.354)
		0.1	1.167	2.686	6.662*	7.010	7.361
		0.3	1.143	2.664	6.465*	6.652	7.336
		0.5	1.096	2.627	6.026	6.043*	7.252
	0.2	0	1.087	2.504	6.367*	6.581	6.852
		0.1	1.084	2.499	6.345*	6.532	6.849
		0.3	1.062	2.478	6.150*	6.207	6.825
		0.5	1.018	2.445	5.611	5.760*	6.748
	5	0	0.7681	1.754	4.315*	4.601	4.786
		0.1	0.7658	1.751	4.300*	4.569	4.786
		0.3	0.7493	1.735	4.159*	4.348	4.769
		0.5	0.7176	1.710	3.838	3.988*	4.714
0.2	0	0	1.132	2.451	3.349*	5.966	6.499
		0.1	1.128	2.444	3.338*	5.919	6.496
		0.3	1.098	2.410	3.240*	5.605	6.473
		0.5	1.041	2.355	3.029*	5.067	6.347
	0.2	0	1.053	2.286	3.192*	5.586	6.070
		0.1	1.049	2.280	3.181*	5.543	6.067
		0.3	1.021	2.249	3.087*	5.249	6.045
		0.5	0.9695	2.199	2.886*	4.745	5.969
	5	0	0.7378	1.570	2.182*	3.757	4.135
		0.1	0.7348	1.566	2.175*	3.728	4.133
		0.3	0.7145	1.542	2.112*	3.529	4.118
		0.5	0.6766	1.503	1.977*	3.184	4.063

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