The 2017 World Congress on
Advances in Structural Engineering and Mechanics (ASEM17)
28 August - 1 September, 2017, Ilsan(Seoul), Korea

Plane coordinate systems with special reference to problems in plasticity: a review

*Sergei Alexandrov[1)] and Alexander Pirumov[2)]

[1)] *Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow 119526, Russia*
[1)] *sergei_alexandrov@spartak.ru*

[2)] *Department of Electrotechnics and Mechanics, Moscow Technological University, Moscow 119526, Russia*

ABSTRACT

Several remarkable properties of characteristic – based and principal line coordinate systems are reviewed or derived for several rigid plastic material models. These properties can be used to simplify solution of boundary value problems. In particular, the R – S method and Mikhlin coordinates are extended to two models of pressure-dependent plasticity. It is shown that methods of solution of boundary value problems developed in the classical theory of metal plasticity can be directly used to solve boundary value problems in pressure-dependent plasticity.

1. INTRODUCTION

The plane strain equations of several rigid plastic models are hyperbolic. Therefore, the characteristic lines can be used as coordinate curves. It is shown that such characteristic – based coordinate systems exhibit remarkable properties that can be used to simplify solution of boundary value problems. Another coordinate system considered in the present paper is a curvilinear orthogonal coordinate system in which the coordinate curves coincide with trajectories of the principal stress directions. This coordinate system also exhibits remarkable properties.

2. CHARACTERISTIC - BASED COORDINATE SYSTEMS

In the classical theory of metal plasticity the system of equations comprising any pressure – independent yield criterion together with the stress equilibrium equations under plane strain conditions forms a statically determinate system and this system is

---

[1)] Professor
[2)] Professor

hyperbolic (Hill, 1950). This is also a property of the system of equations comprising the several pressure - dependent yield criteria together with the stress equilibrium equations under plane strain conditions (Spencer, 1964, Druyanov, 1993). It is therefore natural to use characteristic – based coordinate systems for solving boundary value problems for these models. These coordinate systems exhibit remarkable mathematical properties that can be used for developing efficient methods of solution. In what follows it is assumed that both families of characteristic lines are curved.

2.1 R – S method

In the classical theory of metal plasticity characteristics are orthogonal. The two families of the α – and β – characteristic lines are regarded as a pair of right – handed curvilinear orthogonal axes of reference. By the convention, the line of action of the algebraically greatest principal stress falls in the first and third quadrants. The system of equations comprising any pressure – independent yield criterion together with the stress equilibrium equations is equivalent to

$$p - p_0 = 2k(\beta - \alpha), \quad \psi - \psi_0 = \beta + \alpha. \tag{1}$$

Here p is the mean in-plane pressure, ψ is the angle between the x – axis of an arbitrary Cartesian coordinate system and the direction of the algebraically greatest principal stress measured from the x - axis anticlockwise, k is the shear yield stress, a material constant, p_0 and ψ_0 are constants. The radii of curvature R and S of the α – and β – characteristic lines are defined as

$$\frac{1}{R} = \frac{\partial \psi}{\partial s_\alpha}, \quad \frac{1}{S} = -\frac{\partial \psi}{\partial s_\beta} \tag{2}$$

where $\partial/\partial s_\alpha$ and $\partial/\partial s_\beta$ are space derivatives taken along the α – and β – characteristic lines respectively. The angle between the direction of the algebraically greatest principal stress and each of the characteristic directions is equal to $\pi/4$ (Hill, 1950). Therefore,

$$\frac{\partial x}{\partial s_\alpha} = \cos\left(\psi - \frac{\pi}{4}\right), \quad \frac{\partial x}{\partial s_\beta} = -\sin\left(\psi - \frac{\pi}{4}\right), \quad \frac{\partial y}{\partial s_\alpha} = \sin\left(\psi - \frac{\pi}{4}\right), \quad \frac{\partial y}{\partial s_\beta} = \cos\left(\psi - \frac{\pi}{4}\right) \tag{3}$$

Using Eqs. (1) and (2) it is possible to rewrite Eq. (3) as

$$\frac{\partial x}{\partial \alpha} = R\cos\left(\psi - \frac{\pi}{4}\right), \quad \frac{\partial x}{\partial \beta} = S\sin\left(\psi - \frac{\pi}{4}\right), \quad \frac{\partial y}{\partial \alpha} = R\sin\left(\psi - \frac{\pi}{4}\right), \quad \frac{\partial y}{\partial \beta} = -S\cos\left(\psi - \frac{\pi}{4}\right) \tag{4}$$

The compatibility equations are in

$$\frac{\partial^2 x}{\partial\alpha\partial\beta}=\frac{\partial^2 x}{\partial\beta\partial\alpha},\quad \frac{\partial^2 y}{\partial\alpha\partial\beta}=\frac{\partial^2 y}{\partial\beta\partial\alpha}. \tag{5}$$

Substituting Eq. (4) into Eq. (5) and taking into account Eq. (1) leads to

$$\frac{\partial R}{\partial\beta}\cos\left(\psi-\frac{\pi}{4}\right)-R\sin\left(\psi-\frac{\pi}{4}\right)=\frac{\partial S}{\partial\alpha}\sin\left(\psi-\frac{\pi}{4}\right)+S\cos\left(\psi-\frac{\pi}{4}\right),$$
$$\frac{\partial R}{\partial\beta}\sin\left(\psi-\frac{\pi}{4}\right)+R\cos\left(\psi-\frac{\pi}{4}\right)=-\frac{\partial S}{\partial\alpha}\cos\left(\psi-\frac{\pi}{4}\right)+S\cos\left(\psi-\frac{\pi}{4}\right).$$

Solving these equations for $\partial R/\partial\beta$ and $\partial S/\partial\alpha$ results in

$$\frac{\partial R}{\partial\beta}=S,\quad \frac{\partial S}{\partial\alpha}=-R. \tag{6}$$

It is evident from these equations that *R* and *S* separately satisfy the equation of telegraphy:

$$\frac{\partial^2 R}{\partial\alpha\partial\beta}+R=0,\quad \frac{\partial^2 S}{\partial\alpha\partial\beta}+S=0. \tag{7}$$

Each of these equations can be integrated by the method of Riemann. In particular, the Green's function is the Bessel function of zero order. Once Eq. (6) or Eq. (7) has been solved, the net of characteristics is determined from Eq. (4) by integration. In Eq. (4), the angle ψ should be eliminated by means of Eq. (1). This method of constructing characteristic nets, known as the *R* – *S* method, has been developed by Hill (1950). Using the technique described above the *R* – *S* method has been extended to two models of pressure – dependent plasticity in Alexandrov (2015) and Alexandrov and Lyamina (2015). In particular, the system of equations comprising the Mohr - Coulomb yield criterion together with the stress equilibrium equations has been studied in Alexandrov (2015). It has been shown that the radii of curvature of the characteristic lines defined by Eq. (2) satisfy the following equations:

$$\frac{\partial R}{\partial\beta}+R\sin\phi=S,\quad \frac{\partial S}{\partial\alpha}-S\sin\phi=-R \tag{8}$$

Here ϕ is the angle of external friction. Equation (8) reduces to Eq. (6) at $\phi=0$. Introduce new quantities R_0 and S_0 by the following equations:

$$R=R_0\exp\left[(\alpha-\beta)\sin\phi\right],\quad S=S_0\exp\left[(\alpha-\beta)\sin\phi\right]. \tag{9}$$

Substituting Eq. (9) into Eq. (8) shows that these new quantities separately satisfy the equation of telegraphy:

$$\frac{\partial^2 R_0}{\partial\alpha\partial\beta} + R_0 = 0, \quad \frac{\partial^2 S_0}{\partial\alpha\partial\beta} + S_0 = 0. \tag{10}$$

The system of equations comprising the pyramid yield criterion proposed in Druyanov (1993) together with the stress equilibrium equations has been studied in Alexandrov and Lyamina (2015). It has been shown that the radii of curvature of the characteristic lines defined by Eq. (2) satisfy the following equations:

$$\frac{\partial R}{\partial\beta} + R\cos 2\varphi = S, \quad \frac{\partial S}{\partial\alpha} - S\cos 2\varphi = -R \tag{11}$$

Here

$$\varphi = \arctan\chi, \quad \chi = \frac{3p_s + 2\tau_s}{3p_s - 4\tau_s} \tag{12}$$

where τ_s is the shear yield stress and p_s is the yield stress in hydrostatic compression. Introduce new quantities R_0 and S_0 by the following equations:

$$R = R_0\exp\left[(\alpha-\beta)\cos 2\varphi\right], \quad S = S_0\exp\left[(\alpha-\beta)\cos 2\varphi\right]. \tag{13}$$

Substituting Eq. (13) into Eq. (11) shows that these new quantities separately satisfy Eq. (10).

2.2 Mikhlin coordinates

The Mikhlin coordinates, $\bar{x}$ and $\bar{y}$, are coordinates of a typical point *P* with respect to straight axes passing through a fixed origin *O* and parallel to the characteristic directions at *P*. In the classical theory of plasticity the Mikhlin coordinates are orthogonal and are given by the equations (Hill, 1950):

$$\bar{x} = x\cos\left(\psi - \frac{\pi}{4}\right) + y\sin\left(\psi - \frac{\pi}{4}\right), \quad \bar{y} = -x\sin\left(\psi - \frac{\pi}{4}\right) + y\cos\left(\psi - \frac{\pi}{4}\right) \tag{14}$$

Differentiating the first equation with respect to β and the second with respect to α gives

$$\frac{\partial \bar{x}}{\partial \beta} = \frac{\partial x}{\partial \beta}\cos\left(\psi - \frac{\pi}{4}\right) - x\sin\left(\psi - \frac{\pi}{4}\right) + \frac{\partial y}{\partial \beta}\sin\left(\psi - \frac{\pi}{4}\right) + y\cos\left(\psi - \frac{\pi}{4}\right)$$
$$\frac{\partial \bar{y}}{\partial \alpha} = -\frac{\partial x}{\partial \alpha}\sin\left(\psi - \frac{\pi}{4}\right) - x\cos\left(\psi - \frac{\pi}{4}\right) + \frac{\partial y}{\partial \alpha}\cos\left(\psi - \frac{\pi}{4}\right) - y\sin\left(\psi - \frac{\pi}{4}\right) \qquad (15)$$

It has been taken into account here that ψ satisfies Eq. (1). Also, the α - and β - characteristic curves are determined by the equations:

$$\frac{dy}{dx} = \tan\left(\psi - \frac{\pi}{4}\right), \quad \frac{dy}{dx} = \tan\left(\psi + \frac{\pi}{4}\right)$$

respectively. It follows from these equations that

$$\frac{\partial y}{\partial \alpha} = \tan\left(\psi - \frac{\pi}{4}\right)\frac{\partial x}{\partial \alpha}, \quad \frac{\partial y}{\partial \beta} = \tan\left(\psi + \frac{\pi}{4}\right)\frac{\partial x}{\partial \beta} \qquad (16)$$

Substituting Eq. (16) into Eq. (15) and using Eq. (14) results in

$$\frac{\partial \bar{x}}{\partial \beta} = \bar{y}, \quad \frac{\partial \bar{y}}{\partial \alpha} = -\bar{x}. \qquad (17)$$

It is evident from these equations that $\bar{x}$ and $\bar{y}$ separately satisfy the equation of telegraphy:

$$\frac{\partial^2 \bar{x}}{\partial \alpha \partial \beta} + \bar{x} = 0, \quad \frac{\partial^2 \bar{y}}{\partial \alpha \partial \beta} + \bar{y} = 0. \qquad (18)$$

This result has been extended to the pyramid yield criterion in Alexandrov (2017). In particular, by analogy to Eq. (13) it is possible to introduce new quantities $\bar{X}$ and $\bar{Y}$ as

$$\bar{x} = \bar{X}\exp\left[(\alpha - \beta)\cos\varphi/2\right], \quad \bar{y} = \bar{Y}\exp\left[(\alpha - \beta)\cos\varphi\right] \qquad (19)$$

These new quantities separately satisfy the equation of telegraphy:

$$\frac{\partial^2 \bar{X}}{\partial \alpha \partial \beta} + \bar{X} = 0, \quad \frac{\partial^2 \bar{Y}}{\partial \alpha \partial \beta} + \bar{Y} = 0. \qquad (20)$$

The system of equations comprising the Mohr - Coulomb yield criterion together with the stress equilibrium equations can be treated in a similar manner. As a result, it is possible to show that the quantities $\bar{X}$ and $\bar{Y}$ defined by the equations $\bar{x} = \bar{X}\exp\left[(\alpha-\beta)\sin\phi\right], \quad \bar{y} = \bar{Y}\exp\left[(\alpha-\beta)\sin\phi\right]$ satisfy Eq. (20).

3. PRINCIPAL LINE COORDINATES

Introduce a curvilinear orthogonal coordinate system (μ, ν) whose coordinate curves coincide with trajectories of the principal stress directions. Denoting the principal stresses by σ_μ and σ_ν, respectively, the equilibrium equations may be written as (Malvern, 1969)

$$\frac{\partial(h_\nu \sigma_\mu)}{\partial \mu} - \sigma_\nu \frac{\partial h_\nu}{\partial \mu} = 0, \quad \frac{\partial(h_\mu \sigma_\nu)}{\partial \nu} - \sigma_\mu \frac{\partial h_\mu}{\partial \nu} = 0 \tag{21}$$

Here h_μ and h_ν are the scale factors for the $\mu-$ and $\nu-$ curves, respectively. Since the shear stress component vanishes relative to the chosen coordinate system, any plane strain pressure-independent yield criterion reduces to $\sigma_\mu - \sigma_\nu = K$ where *K* is constant. Substituting this yield criterion into Eq. (21) gives

$$h_\nu \frac{\partial \sigma_\mu}{\partial \mu} + K\frac{\partial h_\nu}{\partial \mu} = 0, \quad h_\mu \frac{\partial \sigma_\mu}{\partial \nu} - K\frac{\partial h_\mu}{\partial \nu} = 0 \tag{22}$$

These equations can be immediately integrated to yield

$$\sigma_\mu = -K\ln\left[\frac{h_\nu}{H_\nu(\nu)}\right], \quad \sigma_\mu = K\ln\left[\frac{h_\mu}{H_\mu(\mu)}\right] \tag{23}$$

Here $H_\nu(\nu)$ is an arbitrary function of ν and $H_\mu(\mu)$ is an arbitrary function of μ. It follows from Eq. (23) that

$$\frac{h_\nu}{H_\nu(\nu)} = \frac{H_\mu(\mu)}{h_\mu}. \tag{24}$$

Different choices of the functions $H_\nu(\nu)$ and $H_\mu(\mu)$ merely change the scale of the $\nu-$ and $\mu-$ curves, respectively. Therefore, without loss of generality it is possible to put $H_\nu(\nu) = H_\mu(\mu) = 1$. Then, Eq. (24) becomes

$$h_\nu h_\mu = 1. \tag{25}$$

This property of the principal line coordinate system has been derived by Sadowsky (1941). This result has been extended to the system of equations comprising the Mohr - Coulomb yield criterion together with the stress equilibrium equations in Alexandrov and Harris (2017). In this case, the scale factors satisfy the following equation:

$$h_\nu^b h_\mu = 1 \tag{26}$$

where

$$b = \frac{m + \sin\phi}{m - \sin\phi} \tag{27}$$

and $m = 1$ if $\sigma_\mu > \sigma_\nu$ and $m = -1$ if $\sigma_\mu < \sigma_\nu$. It is evident that Eq. (26) reduces to Eq. (25) if $\phi = 0$.

4. CONCLUSIONS

On the assumption of plane strain conditions the system of equations comprising a yield criterion and the equilibrium equations has been studied in a characteristic based coordinate system and in a curvilinear orthogonal coordinate system in which the coordinate curves coincide with trajectories of the principal stress directions. Several yield criteria have been considered. Remarkable properties of both coordinate systems have been reviewed or derived.

REFERENCES

Alexandrov, S. (2015), “Geometry of plane strain characteristic fields in pressure-dependent plasticity,” *ZAMM,* **95**(11), 1296-1301.

Alexandrov, S. (2017), “Riemann's method in modelling of powder metallurgy processes,” Proc. *CIRP,* in press.

Alexandrov, S. and Harris, D. (2017), “Geometry of principal stress trajectories for a Mohr-Coulomb material under plane strain,” *ZAMM,* **97**(4), 473-476.

Alexandrov, S. and Lyamina, E. (2015), “Riemann method for the plane strain of a homogeneous porous plastic material,” *Mech. Solids,* **50**(2), 171-175.

Druyanov, B. (1993), “Technological mechanics of porous bodies,” New York, Clarendon Press.

Hill, R. (1950), “The mathematical theory of plasticity,” Oxford, Oxford University Press.

Malvern, L.E. (1969), “Introduction to the mechanics of a continuum medium,” Englewood Cliffs, Prentice-Hall.

Sadowsky, M.A. (1941), “Equiareal pattern of stress trajectories in plane plastic strain”, *J. Appl. Mech., ASME,* **63**, A74-A76.

Spencer, A.J.M. (1964), “A theory of the kinematics of ideal soil under plane strain conditions”, *J. Mech. Phys. Solids,* **12**, 337-351.