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Penetration of a spherical conductive punch into a piezoelectric inhomogeneous half-space

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ABSTRACT

Contact problem on indentation of an electro-elastic piezoelectric functionally graded (FG) half-space by a rigid spherical punch is considered. The half-space consists of a FG layer (coating) with arbitrary variation of electromechanical properties in depth and a homogeneous semi-infinite substrate. The punch is assumed to be an ideal electric conductor. Normal centrally applied force and constant electric charge are applied to the punch, leading to electro-elastic deformation of the half-space. The problem was described mathematically in terms of the linear theory of electro-elasticity and reduced to the solution of a system of integral equations. The bilateral asymptotic method was used to construct approximated solution of that system. Analytical expressions for contact stresses, electric induction, indentation force and electric charge coatings are provided. The results obtained are asymptotically exact both for thin and thick coatings and of high accuracy for intermediate thickness of the coating.

1. INTRODUCTION

Many researchers have carried out investigations for the contact interaction of elastic solids with homogeneous or FG coatings, see Alinia et al. (2016), Lu et al. (2008) for example. Su et al. (2016), Ma et al. (2014) and Liu et al. (2012) consider contact of a rigid punch and electroelastic, magnetoelectroelastic or thermoelastic half-plane (or half-space) with FG coatings. They use a model of piecewise constant or exponential variation of properties in the coating. The problems were reduced to singular integral equations which were solved numerically by the collocation method. This method efficiently works for intermediate thickness of the coating but does not allow to obtain a

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solution effective for thin coatings that are most important for applications.

The present paper addresses the axisymmetric contact of the rigid spherical punch and an electroelastic half-space with FG coating. In contrary to the abovementioned results, an arbitrary variation of the properties in the coating is considered. The solution of the problem is provided in the analytical form that is asymptotically exact for thin and thick coatings and of high accuracy for the intermediate thickness of the coating.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Let us consider electroelastic half-space consisting of a FG piezoelectric layer (coating) of thickness *H*, and a homogeneous piezoelectric half-space (substrate). The axis of symmetry coincides with the direction of pre-polarization field. Cylindrical coordinate system *r*, φ, *z* is chosen with the *z* axis being the axis of symmetry of electromechanical properties. Linear constitutive equations for a piezoelectric material have the following form:

$$\begin{aligned}
&\sigma_r = c_{11}\frac{\partial u}{\partial r} + c_{12}\frac{u}{r} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \psi}{\partial z}; \quad && \sigma_\varphi = c_{12}\frac{\partial u}{\partial r} + c_{11}\frac{u}{r} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \psi}{\partial z};\\
&\sigma_z = c_{13}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + c_{33}\frac{\partial w}{\partial z} + e_{33}\frac{\partial \psi}{\partial z}; && \tau_{rz} = c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + e_{15}\frac{\partial \psi}{\partial r};\\
&D_r = -\varepsilon_{11}\frac{\partial \psi}{\partial r} + e_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right); && D_z = -\varepsilon_{33}\frac{\partial \psi}{\partial z} + e_{31}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + e_{33}\frac{\partial w}{\partial z}.
\end{aligned} \tag{1}$$

Elastic moduli c_{11}, c_{12}, c_{13}, c_{33}, c_{44}, piezoelectric constants e_{31}, e_{15}, e_{33} and dielectric permeabilities ε_{11}, ε_{33} of the half-space vary with depth as

$$\{c_{kj}, e_{kj}, \varepsilon_{kj}\} = \begin{Bmatrix} c_{kj}^{(c)}(z), e_{kj}^{(c)}(z), \varepsilon_{kj}^{(c)}(z), & -H \le z \le 0 \\ c_{kj}^{(s)}, \quad e_{kj}^{(s)}, \quad \varepsilon_{kj}^{(s)}, & -\infty < z < -H \end{Bmatrix}, \tag{2}$$

where $c_{kj}^{(c)}(z), e_{kj}^{(c)}(z), \varepsilon_{kj}^{(c)}(z)$ are continuously differentiable functions, $c_{kj}^{(s)}, e_{kj}^{(s)}, \varepsilon_{kj}^{(s)}$ are constants. Hereafter, superscripts (*c*) and (*s*) correspond to the coating and to the substrate, respectively. The coating and the substrate are assumed to be glued without sliding:

$$z = -H: \quad \begin{aligned} & w^{(c)} = w^{(s)}, u^{(c)} = u^{(s)}, \psi^{(c)} = \psi^{(s)};\\ & \sigma_z^{(c)} = \sigma_z^{(s)}, \tau_{rz}^{(c)} = \tau_{rz}^{(s)}, D_z^{(c)} = D_z^{(s)}. \end{aligned} \tag{3}$$

Let us consider a rigid spherical punch of radius *R* acting on the boundary of the half-space in the region *z*=0, *x*≤*a* (see Fig. 1). Outside this region the surface is free of stress and electrically insulated. The punch is assumed to be an ideal conductor. The punch is subjected to the normal centrally applied force *P* and constant electric charge

Q. Friction between the punch and the half-space is neglected. Under the action of applied electromechanical loading the punch moves a distance δ downward the *z*-axis and an electrostatic field with potential ψ_0 is formed. Therefore, the boundary conditions take the form:

$$z=0:\ \tau_{rz}^{(c)}=0;\begin{cases}\sigma_z^{(c)}=D_z^{(c)}=0, & r>a;\\ w^{(c)}=-\delta+r^2/2R,\ \psi^{(c)}=-\psi_0 & r\le a.\end{cases} \tag{4}$$

The quantities of primary interest are the contact normal stresses and electric normal induction under the punch:

$$\sigma_z\big|_{z=0}=-p(r),\ D_z\big|_{z=0}=-q(r),\ r\le a; \tag{5}$$

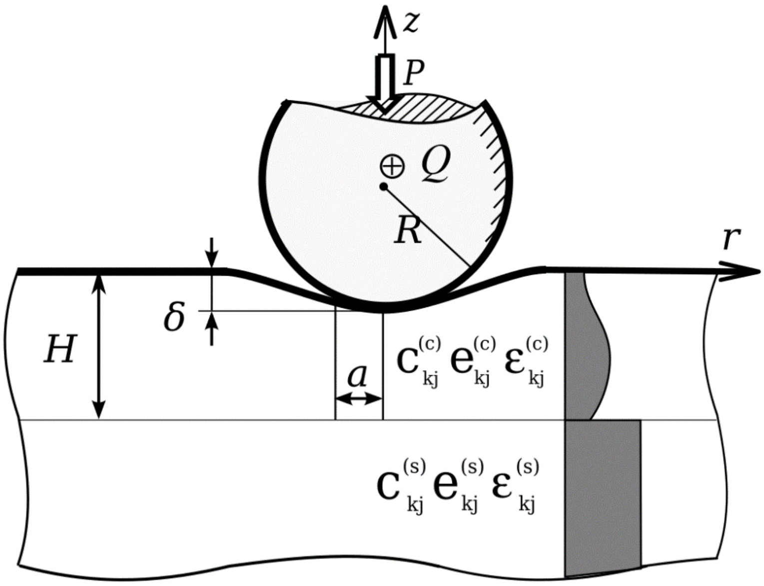

Fig. 1 Statement of the problem

3. SOLUTION OF THE PROBLEM

Let us use the Hankel transformations:

$$u(r,z)=-\int_0^\infty \overline{u}(\gamma,z)\mathrm{J}_1(\gamma r)\gamma d\gamma,\quad \{w,\psi,p,q\}(r,z)=\int_0^\infty\{\overline{w},\overline{\psi},\overline{p},\overline{q}\}(\gamma,z)\mathrm{J}_0(\gamma r)\gamma d\gamma, \tag{6}$$

and let us represent Hankel images of the displacements and electric potential in the following linear combination:

$$\{\overline{u}^{(c)},\overline{w}^{(c)},\overline{\psi}^{(c)}\}(\gamma,z)=-\Theta_{31}^{-1}L_{\{1,3,5\}1}(\gamma,z)\gamma^{-1}\overline{p}(\gamma)-\Theta_{32}^{-1}L_{\{1,3,5\}2}(\gamma,z)\gamma^{-1}\overline{q}(\gamma) \tag{7}$$

where constants Θ_{kj} characterize effective electromechanical properties on the surface of the coating, L_{kj} are the compliance functions to be determined from the solution of two-point boundary value problem for a system of ordinary differential equations with variable coefficients, see Vasiliev et al. (2016) for details.

Similar to Aizikovich and Aleksandrov (1982) the following asymptotic behavior of the compliance functions can be obtained:

$$\begin{aligned} &L_{kj}(\gamma) = \Theta_{kj} / \Theta_{kj}^{s} + b\gamma + c\gamma^2 + \mathrm{O}(\gamma^3), \gamma \to 0 \\ &L_{kj}(\gamma) = 1 + \tilde{b}\gamma^{-1} + \tilde{c}\gamma^{-2} + \mathrm{O}(\gamma^{-3}), \gamma \to \infty \end{aligned} \quad (8)$$

where constants Θ_{kj}^{s} characterize effective electromechanical properties of the substrate. Using Eq. (4), (6) and (7) the following system of dual integral equations can be obtained:

$$\begin{aligned} &\int_0^a p(\rho)\rho d\rho \int_0^\infty \frac{L_{31}(\gamma,0)}{\Theta_{31}} \mathrm{J}_0(\gamma\rho)\mathrm{J}_0(\gamma r)d\gamma + \int_0^a q(\rho)\rho d\rho \int_0^\infty \frac{L_{32}(\gamma,0)}{\Theta_{32}} \mathrm{J}_0(\gamma\rho)\mathrm{J}_0(\gamma r)d\gamma = \delta - \frac{r^2}{2R} \\ &\int_0^a p(\rho)\rho d\rho \int_0^\infty \frac{L_{51}(\gamma,0)}{\Theta_{51}} \mathrm{J}_0(\gamma\rho)\mathrm{J}_0(\gamma r)d\gamma + \int_0^a q(\rho)\rho d\rho \int_0^\infty \frac{L_{52}(\gamma,0)}{\Theta_{52}} \mathrm{J}_0(\gamma\rho)\mathrm{J}_0(\gamma r)d\gamma = \psi_0 \end{aligned} \quad (9)$$

It should be pointed out that $\Theta_{51}=\Theta_{32}$, $L_{51}(\gamma,z)=L_{32}(\gamma,z)$, $\Theta_{52}<0$, $\Theta_{32}>0$, $\Theta_{31}>0$, $L_{kj}(\gamma,z)>0$.

To solve the system of integral equations (9) with kernel transforms L_{kj} possessing properties (8) the bilateral asymptotic method, see Aizikovich (1990), can be used. It is based on an idea of approximation of the compliance functions by the ratio of fractional quadratic functions:

$$\begin{aligned} &L_{kj}(\lambda\gamma) \approx \Pi_{kj}(\lambda\gamma) = P_{kj1}(\lambda^2\gamma^2) / P_{kj2}(\lambda^2\gamma^2) \\ &P_{kj1}(\lambda^2\gamma^2) = \prod_{i=1}^{N_{kj}} (\lambda^2\gamma^2 + A_{kji}^2), P_{kj2}(\lambda^2\gamma^2) = \prod_{i=1}^{N_{kj}} (\lambda^2\gamma^2 + B_{kji}^2), A_{kji} \neq B_{slm} \end{aligned} \quad (10)$$

Using approximation (10) an approximated analytical solution of the system (9) can be obtained similar to Volkov et al. (2017):

$$p(r) = \frac{4a\Theta^{(s)}}{\pi R\Theta_{52}^{(s)}} \left[\sqrt{1-r^2} + \frac{R}{2a^2\lambda} \sum_{j=1}^{N} C_j \omega_j \frac{\Pi_{52}(i\omega_j)}{\Pi_{52}(0)} \int_r^1 \frac{\mathrm{sh}(\omega_j \lambda^{-1} t) dt}{\sqrt{t^2 - r^2}} \right] \quad (11)$$

$$q(r)=\frac{2\Theta^{(s)}}{\pi a\Theta_{32}^{(s)}}\left[\left(\frac{\Theta_{32}^{(s)}}{\Theta_{31}^{(s)}}\psi_0-\delta\right)\frac{1}{\sqrt{1-r^2}}+\sum_{j=1}^{N}C_j\frac{\Pi_{32}(i\omega_j)}{\Pi_{32}(0)}\left(\frac{\mathrm{ch}\,\omega_j\lambda^{-1}}{\sqrt{1-r^2}}-\frac{\omega_j}{\lambda}\int_r^1\frac{\mathrm{sh}(\omega_j\lambda^{-1}t)dt}{\sqrt{t^2-r^2}}\right)\right]-$$
$$-\frac{4\Theta^{(s)}a}{\pi\Theta_{32}^{(s)}R}\left(\sqrt{1-r^2}-\frac{1+2\lambda^2E_2}{2\sqrt{1-r^2}}\right) \quad (12)$$

The dimensionless variables were used above (primes are omitted):

$$\{\lambda,r',\rho'\}=\{H,r,\rho\}/a,\ \{p',q'\}(\rho')=\{p,q\}(\rho'a),\ L'_{kj}(\gamma)=L_{kj}(\gamma/H,0) \quad (13)$$

Constants C_j are the solution of the system of algebraic equations:

$$\begin{aligned}
&\sum_{j=1}^{N_m}C_j\frac{\Pi_{52}(i\omega_j)}{\Pi_{52}(0)}\frac{B_{31n}F\left(B_{31n},\omega_j,\omega_j\lambda^{-1}\right)}{B_{31n}^2-\omega_j^2}=\delta-\frac{\Theta_{52}^{(s)}}{\Theta_{32}^{(s)}}\psi_0-\frac{a^2}{R}\left(1+\frac{2\lambda}{B_{31n}}+2\lambda^2\left(\frac{1}{B_{31n}^2}+E_1\right)\right),\\
&\sum_{j=1}^{N_m}C_j\frac{\Pi_{52}(i\omega_j)}{\Pi_{52}(0)}\frac{B_{32n}F\left(B_{32n},\omega_j,\omega_j\lambda^{-1}\right)}{B_{32n}^2-\omega_j^2}=\delta-\frac{\Theta_{52}^{(s)}}{\Theta_{32}^{(s)}}\psi_0-\frac{a^2}{R}\left(1+\frac{2\lambda}{B_{32n}}+2\lambda^2\left(\frac{1}{B_{32n}^2}+E_1\right)\right)\\
&\sum_{j=1}^{N_m}C_j\frac{\Pi_{32}(i\omega_j)}{\Pi_{32}(0)}\frac{B_{32n}F\left(B_{32n},\omega_j,\omega_j\lambda^{-1}\right)}{B_{32n}^2-\omega_j^2}=\delta-\frac{\Theta_{32}^{(s)}}{\Theta_{31}^{(s)}}\psi_0-\frac{a^2}{R}\left(1+\frac{2\lambda}{B_{32n}}+2\lambda^2\left(\frac{1}{B_{32n}^2}+E_2\right)\right),\\
&\sum_{j=1}^{N_m}C_j\frac{\Pi_{32}(i\omega_j)}{\Pi_{32}(0)}\frac{B_{52n}F\left(B_{52n},\omega_j,\omega_j\lambda^{-1}\right)}{B_{52n}^2-\omega_j^2}=\delta-\frac{\Theta_{32}^{(s)}}{\Theta_{31}^{(s)}}\psi_0-\frac{a^2}{R}\left(1+\frac{2\lambda}{B_{52n}}+2\lambda^2\left(\frac{1}{B_{52n}^2}+E_2\right)\right)
\end{aligned} \quad (14)$$

Constants ω_j are the roots of the characteristic equation:

$$\Pi_{32}^2(i\omega_j)\Theta_{32}^{-2}-\Pi_{52}(i\omega_j)\Pi_{31}(i\omega_j)\Theta_{52}^{-1}\Theta_{31}^{-1}=0. \quad (15)$$

Following notations were used above:

$$N=N_{31}+2N_{32}+N_{52},\ F(C,D,\omega)=C\mathrm{ch}\omega+D\mathrm{sh}\omega,\ \Theta^{(s)}=\frac{\Theta_{31}^{(s)}\Theta_{52}^{(s)}\Theta_{32}^{(s)2}}{\Theta_{32}^{(s)2}-\Theta_{31}^{(s)}\Theta_{52}^{(s)}} \quad (16)$$

$$\begin{aligned}
E_1&=\sum_{j=1}^{N_{31}}(A_{31j}^{-2}-B_{31j}^{-2})+\frac{\Theta^{(s)}}{\Theta_{32}^{(s)2}}\left[\sum_{j=1}^{N_{31}}(A_{31j}^{-2}-B_{31j}^{-2})-2\sum_{j=1}^{N_{32}}(A_{32j}^{-2}-B_{32j}^{-2})+\sum_{j=1}^{N_{52}}(A_{52j}^{-2}-B_{52j}^{-2})\right]\\
E_2&=\sum_{j=1}^{N_{32}}(A_{32j}^{-2}-B_{32j}^{-2})+\frac{\Theta^{(s)}}{\Theta_{52}^{(s)}\Theta_{31}^{(s)}}\left[\sum_{j=1}^{N_{31}}(A_{31j}^{-2}-B_{31j}^{-2})-2\sum_{j=1}^{N_{32}}(A_{32j}^{-2}-B_{32j}^{-2})+\sum_{j=1}^{N_{52}}(A_{52j}^{-2}-B_{52j}^{-2})\right]
\end{aligned} \quad (17)$$

Indentation force and electric charge are obtained from the conditions of equilibrium of the punch $P=2\pi a^2\int_0^1 rp(r)dr$ and $Q=2\pi a^2\int_0^1 rq(r)dr$ in the form:

$$P = \frac{8\pi a^3 \Theta^{(s)}}{3R\Theta_{52}^{(s)}} \left[1 + \frac{3R}{2a^2\lambda} \sum_{j=1}^{N} C_j \omega_j \frac{\Pi_{52}(i\omega_j)}{\Pi_{52}(0)} \left(\mathrm{ch}\frac{\omega_j}{\lambda} - \frac{\lambda}{\omega_j} \mathrm{sh}\frac{\omega_j}{\lambda} \right) \right] \quad (18)$$

$$Q = -\frac{4\Theta^{(s)} a}{\Theta_{32}^{(s)}} \left(\delta - \psi_0 \frac{\Theta_{32}^{(s)}}{\Theta_{31}^{(s)}} - \lambda \sum_{j=1}^{N} C_j \frac{\Pi_{32}(i\omega_j)}{\Pi_{32}(0)} \frac{\mathrm{sh}(\omega_j \lambda^{-1})}{\omega_j} - \frac{a^2}{R} \left(\frac{1}{3} + 2\lambda^2 E_2 \right) \right) \quad (19)$$

The unknown radius of the contact area satisfies the following equation:

$$\delta - \frac{\Theta_{52}^{(s)}}{\Theta_{32}^{(s)}} \psi_0 - \sum_{j=1}^{N} C_j \frac{\Pi_{52}(i\omega_j)}{\Pi_{52}(0)} \mathrm{ch}(\frac{\omega_j a}{H}) - \frac{a^2}{R} - \frac{2H^2}{R} E_2 = 0 \quad (20)$$

4. CONCLUSIONS

Analytical form of the solution (11)-(20) is suitable for qualitative analysis of the main characteristics of the electroelastic contact such as contact stresses, electric induction, indentation force and electric charge. One can follow on dependence of these quantities on the basic parameters of the problem such as indentation depth δ, electric potential ψ_0 and relative thickness of the coating λ.

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