

equations for strains. Therefore, the original flow theory of plasticity reduces to the corresponding deformation theory of plasticity. A number of solutions for the deformation theory of plasticity used in conjunction with the von Mises yield criterion are also available (Eraslan 2002, Eraslan and Akis 2003, You *et al.* 2000, Hojjati and Hassani 2008). The stress distributions in rotating disks from von Mises and Tresca yield criteria have been compared in Rees (1999). It has been shown that the choice of the yield criterion may affect the final result. It is generally accepted that the deformation theory of plasticity is valid only when dealing with proportional loadings. On the other hand, it is known that the strain path is in general not proportional in thin disks (Pirumov *et al.* 2013). It is therefore of interest to find the distribution of stresses and strains in thin rotating disks using the flow theory of plasticity. Finite difference solutions for such material models have been given in Alexandrova *et al.* (2004) and Alexandrova (2012). Recently, an approach to find a semi-analytic solution for the von Mises yield criterion and its associated flow rule has been proposed in Lomakin *et al.* (2016). An advantage of this approach is that the original boundary value problems in two independent variables is reduced to solving several ordinary differential equations (these equations can be solved one by one) and to evaluating ordinary integrals. In the present paper, the approach developed in Lomakin *et al.* (2016) is extended to the anisotropic yield criterion proposed in Hill (1950).

2. STATEMENT OF THE PROBLEM

Consider a thin annular disk of outer radius b_0 and inner radius a_0 rotating with an angular velocity ω about its axis. The thickness of the disk is constant. Strains are supposed to be infinitesimal. The disk has no stress at $\omega=0$. It is natural to introduce a cylindrical coordinate system (r, θ, z) with the z-axis coinciding with the axis of symmetry of the disk. Let σ_r , σ_θ and σ_z be the normal stresses relative to this coordinate system. These stresses are the principal stresses. The boundary value problem is axisymmetric, and its solution is independent of θ . The circumferential displacement vanishes everywhere. The state of stress in the rotating disk is two-dimensional ($\sigma_z = 0$). The angular velocity ω slowly increases from zero to some prescribed value. Therefore, the component of the acceleration vector in the circumferential direction is neglected. The boundary conditions are

$$\sigma_r = 0 \tag{1}$$

for $r = a_0$ and $r = b_0$. In general, the disk consists of two regions, elastic and plastic. The elastic strains are related by Hooke's law to the stresses. In the case under consideration this law in the cylindrical coordinate system reads

$$\varepsilon_r^e = \frac{\sigma_r - \nu\sigma_\theta}{E}, \quad \varepsilon_\theta^e = \frac{\sigma_\theta - \nu\sigma_r}{E}, \quad \varepsilon_z^e = -\frac{\nu(\sigma_r + \sigma_\theta)}{E}. \quad (2)$$

Here ν is Poisson's ratio and E is Young's modulus. The superscript e denotes the elastic part of the strain and will denote the elastic part of the strain rate as well. In the elastic region, the whole strain is elastic. The superscript e is employed in Eq. (2) as the same equations are satisfied by the elastic part of the strain in the plastic region. The superscript can be dropped in the elastic region. It is assumed that the anisotropic yield criterion proposed in Hill (1950) and its associated flow rule are valid in the plastic region. This yield criterion is written in plane stress as (Alexandrova and Alexandrov 2004)

$$\sigma_r^2 + p_\theta^2 - \eta\sigma_r p_\theta = \sigma_0^2, \quad (3)$$

where

$$p_\theta = \frac{\sigma_\theta}{\eta_1}, \quad \eta = 2HXY, \quad \eta_1 = \frac{Y}{X}, \quad \sigma_0 = \frac{1}{\sqrt{G+H}}.$$

Moreover,

$$2F = \frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2}, \quad 2G = \frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2}, \quad 2H = \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}$$

where X , Y and Z be the yield stresses in the r -, θ - and z - directions, respectively. Let $\dot{\varepsilon}_r^p$, $\dot{\varepsilon}_\theta^p$ and $\dot{\varepsilon}_z^p$ be the plastic strain rates. The associated flow rule under plane stress conditions can be written as (Alexandrov 2015)

$$\begin{aligned} \dot{\varepsilon}_r^p &= \lambda(H+G)\left(\sigma_r - \frac{\eta}{2}p_\theta\right), & \dot{\varepsilon}_\theta^p &= \lambda\left(p_\theta - \frac{\eta}{2}\sigma_r\right), \\ \dot{\varepsilon}_z^p &= -\lambda(H+G)\left[\left(1 - \frac{\eta}{2\eta_1}\right)\sigma_r + \left(\frac{1}{\eta_1} - \frac{\eta}{2}\right)p_\theta\right]. \end{aligned} \quad (4)$$

where λ is a non-negative multiplier. The superimposed dot denotes the time derivative at fixed r and the superscript p denotes the plastic part of the strain rate and will denote the plastic part of the strain. The total strains and strain rates in the plastic region are

$$\begin{aligned}\varepsilon_r &= \varepsilon_r^e + \varepsilon_r^p, & \varepsilon_\theta &= \varepsilon_\theta^e + \varepsilon_\theta^p, & \varepsilon_z &= \varepsilon_z^e + \varepsilon_z^p, \\ \dot{\varepsilon}_r &= \dot{\varepsilon}_r^e + \dot{\varepsilon}_r^p, & \dot{\varepsilon}_\theta &= \dot{\varepsilon}_\theta^e + \dot{\varepsilon}_\theta^p, & \dot{\varepsilon}_z &= \dot{\varepsilon}_z^e + \dot{\varepsilon}_z^p.\end{aligned}\quad (5)$$

The constitutive equations should be supplemented with the equilibrium equation of the form

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = -\zeta \omega^2 r. \quad (6)$$

Here ζ is the density of the material.

It is convenient to introduce the following dimensionless quantities

$$\rho = \frac{r}{b_0}, \quad \Omega = \frac{\sqrt{3}\zeta\omega^2 b_0^2}{\sigma_0}, \quad a = \frac{a_0}{b_0}, \quad k = \frac{\sigma_0}{E}. \quad (7)$$

The material model adopted is rate-independent. Therefore, the time derivative can be replaced with the derivative with respect to any monotonically increasing parameter. In particular, it is convenient to introduce the following quantities

$$\begin{aligned}\xi_r &= \frac{\partial \varepsilon_r}{\partial \Omega}, & \xi_\theta &= \frac{\partial \varepsilon_\theta}{\partial \Omega}, & \xi_z &= \frac{\partial \varepsilon_z}{\partial \Omega}, \\ \xi_r^e &= \frac{\partial \varepsilon_r^e}{\partial \Omega}, & \xi_\theta^e &= \frac{\partial \varepsilon_\theta^e}{\partial \Omega}, & \xi_z^e &= \frac{\partial \varepsilon_z^e}{\partial \Omega}, \\ \xi_r^p &= \frac{\partial \varepsilon_r^p}{\partial \Omega}, & \xi_\theta^p &= \frac{\partial \varepsilon_\theta^p}{\partial \Omega}, & \xi_z^p &= \frac{\partial \varepsilon_z^p}{\partial \Omega}.\end{aligned}\quad (8)$$

The equation of strain rate compatibility is equivalent to

$$\rho \frac{\partial \xi_\theta}{\partial \rho} = \xi_r - \xi_\theta. \quad (9)$$

Using Eq.(7), Eq. (6) can be transformed to

$$\frac{\partial \sigma_r}{\sigma_0 \partial \rho} + \frac{\sigma_r - \sigma_\theta}{\sigma_0 \rho} = -\frac{\Omega \rho}{\sqrt{3}}. \quad (10)$$

3. SOLUTION

3.1. Purely Elastic Solution

The purely elastic solution of the boundary value problem under consideration is well known (see, for example, Timoshenko and Goodier 1970). Using Eq. (7) the solution satisfying the boundary condition (1) at $\rho=1$ can be written as

$$\begin{aligned}
 \frac{\sigma_r}{\sigma_0} &= A \left(\frac{1}{\rho^2} - 1 \right) + \frac{\Omega(3+\nu)}{8\sqrt{3}} (1 - \rho^2), \\
 \frac{\sigma_\theta}{\sigma_0} &= -A \left(\frac{1}{\rho^2} + 1 \right) + \frac{\Omega(1+3\nu)}{8\sqrt{3}} \left(\frac{3+\nu}{1+3\nu} - \rho^2 \right), \\
 \frac{\varepsilon_r}{k} &= \frac{24A [1+\nu - (1-\nu)\rho^2] + \sqrt{3}\Omega(1-\nu) [3+\nu - 3(1+\nu)\rho^2] \rho^2}{24\rho^2}, \\
 \frac{\varepsilon_\theta}{k} &= \frac{-24A [1+\nu + (1-\nu)\rho^2] + \sqrt{3}\Omega(1-\nu) [3+\nu - (1+\nu)\rho^2] \rho^2}{24\rho^2}, \\
 \frac{\varepsilon_z}{k} &= \frac{\nu}{12} \left\{ 24A - \sqrt{3}\Omega [3+\nu - 2(1+\nu)\rho^2] \right\}.
 \end{aligned} \tag{11}$$

Here A is a constant of integration. Using the boundary condition (1) at $\rho=a$ this constant is determined as

$$A = -\frac{\Omega(3+\nu)a^2}{8\sqrt{3}}. \tag{12}$$

Substituting Eq. (12) into Eq. (11) supplies the distribution of the stresses and strains in the purely elastic disk. In particular,

$$\frac{\sigma_r}{\sigma_0} = 0, \quad \frac{\sigma_\theta}{\sigma_0} = \frac{\Omega [3+\nu + a^2(1-\nu)]}{4\sqrt{3}} \tag{13}$$

at $\rho=a$. Substituting Eq. (13) into the yield criterion (3) shows that

$$\Omega_e = \frac{12\sqrt{3}}{(3+\alpha) [3+\nu + a^2(1-\nu)]} \tag{14}$$

where Ω_e is the value of Ω at which the plastic region starts to develop from the inner radius of the disk. In what follows, it is assumed that $\Omega > \Omega_e$.

The solution (11) is also valid in the elastic region of the elastic/plastic disk. However, A is not given by Eq. (12).

3.2. Elastic/Plastic Stress Solution

The elastic/plastic stress solution is available (Alexandrova and Alexandrov 2004). For completeness, this solution is outlined below. In particular, the stress solution in the plastic region is

$$\sigma_r/\sigma_0 = 2 \cos \psi / \sqrt{4 - \eta^2}, \quad p_\theta/\sigma_0 = \eta \cos \psi / \sqrt{4 - \eta^2} + \sin \psi \quad (15)$$

where ψ is a new function of ρ and Ω . This function should be found from the following equation:

$$\frac{2 \sin \psi}{\sqrt{4 - \eta^2}} \frac{d\psi}{d\rho} - \left[\frac{2F \cos \psi}{(H + F)\sqrt{4 - \eta^2}} - \eta_1 \sin \psi \right] \frac{1}{\rho} - \Omega \rho = 0 \quad (16)$$

The boundary condition to this equation is $\psi = \pi/2$ at $\rho = q$. The solution in the elastic region is

$$\begin{aligned} \frac{\sigma_r}{\sigma_0} &= \frac{B}{\sigma_0} \left(\frac{1}{\rho^2} - 1 \right) + \frac{3 + \nu}{8} \Omega (1 - \rho^2), \\ \frac{\sigma_\theta}{\sigma_0} &= -\frac{B}{\sigma_0} \left(\frac{1}{\rho^2} + 1 \right) + \frac{1 + 3\nu}{8} \Omega \left(\frac{3 + \nu}{1 + 3\nu} - \rho^2 \right) \end{aligned} \quad (17)$$

where B is an arbitrary constant. For a given angular velocity the magnitudes of the radius of the elastic-plastic boundary and B can be determined from the condition of continuity of the stresses across the elastic-plastic boundary.

3.3. Elastic/Plastic Strain Solution

The strain solution in the elastic region follows from Eq. (11). Eliminating λ in Eq.(4), replacing the time derivative with the derivative with respect to Ω and using Eq. (8) lead to

$$\frac{\xi_r^p}{\xi_\theta^p} = \frac{(H+G)(2\sigma_r - \eta p_\theta)}{(2p_r - \eta\sigma_r)}, \quad \frac{\xi_z^p}{\xi_\theta^p} = -(H+G) \frac{[(2\eta_1 - \eta)\sigma_r + (2 - \eta\eta_1)p_\theta]}{\eta_1(2p_r - \eta\sigma_r)}. \quad (18)$$

The stresses in these equations can be eliminated by means of Eq. (15) giving the right hand sides of the equations in Eq. (18) as functions of ψ . These functions can be substituted into Eq. (9) to arrive at the ordinary differential equation for ψ as a function of ρ . This equation should be solved numerically.

4. CONCLUSIONS

This article presents a semi-analytic solution for the stresses and strains within a rotating elastic/plastic annular disk. The yield criterion proposed in Hill (1950) and its associated flow rule have been adopted. Therefore, in contrast to available solutions, the equations to be solved involve strain rates rather than strains. This greatly adds to the difficulties of the solution. The method proposed in Lomakin *et al.* (2016) has been used to facilitate analysis.

ACKNOWLEDGMENT

The research described in this paper has been supported by the grant RFBR-17-51-52001 (Russia).

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