2.The dynamic stress reponse

The Von Mises stress in the three-dismensional space can be defined as follow:

$$\sigma_{vm}^{k\ 2} = \sigma_x^{k2} + \sigma_y^{k2} + \sigma_z^{k2} - \sigma_x^2\sigma_y^2 - \sigma_x^2\sigma_z^2 - \sigma_y^2\sigma_z^2 + 3(\tau_{xy}^{k\ 2} + \tau_{xz}^{k\ 2} + \tau_{yz}^{k\ 2}) \qquad (1)$$

Here, $\{\sigma\}^k = \{\sigma_x^k, \sigma_y^k, \sigma_z^k, \tau_{xy}^k, \tau_{xz}^k, \tau_{yz}^k\}^T$ represent the stress vector of the k-th element of the structure,Equation(1) can be expressed as:

$$\sigma_{vm}^{k\ 2} = \{\sigma\}^T A\{\sigma\} = Trace\{A\left[\{\sigma\}\{\sigma\}^T\right]\} \qquad (2)$$

where A is a symmetric matrix,and $A = \begin{pmatrix} 1 & -0.5 & -0.5 & & & \\ -0.5 & 1 & -0.5 & & & \\ -0.5 & -0.5 & 1 & & & \\ & & & 3 & & \\ & & & & 3 & \\ & & & & & 3 \end{pmatrix}$.

Therefore, the mean square response of the Von Mises stress can be written as:

$$E(\sigma_{vm}^{k\ 2}) = E(\{\sigma\}^T A\{\sigma\}) = E(Trace\{A\left[\{\sigma\}^k\{\sigma\}^{kT}\right]\}) = Trace\{AE\left[\{\sigma\}^k\{\sigma\}^{kT}\right]\} \qquad (3)$$

Where $E\left[\{\sigma\}^k\{\sigma\}^{kT}\right]$ is the covariance matrix of the stress vector $\{\sigma\}^k$.In a finite element analysis,the stress vector $\{\sigma\}^k$ of the k-th element can be expressed as:

$$\{\sigma\}^k = \frac{1}{N_h}\sum_{h=1}^{N_h}\left([D]^k[B]_h^k\right)\{u\}^k = S^k\{u\}^k \qquad (4)$$

where $[D]^k$ is the elastic matrix of the k-th element, $[B]_h^k$ is the strain matrix at the h-th Guass integration point of the k-th element, N_h is the total number for the Guass integration point,and $\{u\}^k$ is nodal displacement vector of the k-th element.Then the covariance matrix $E\left[\{\sigma\}^k\{\sigma\}^{kT}\right]$ can be expressed as:

$$\begin{aligned} E\left[\{\sigma\}^k\{\sigma\}^{kT}\right] &= E\left[(\{S\}^k\{u\}^k)(\{S\}^k\{u\}^k)^T\right] \\ &= E\left[(\{S\}^k\{u\}^k\{u\}^{kT}\{S\}^{kT})\right] \\ &= \{S\}^k E[\{u\}^k\{u\}^{kT}]\{S\}^{kT} \end{aligned} \qquad (5)$$

where $E[\{u\}^k \{u\}^{kT}]$ is the the covariance matrix of the nodal displacement vector $\{u\}^k$.

The finite element equation for a structural dynamic problem can be expressed as:

$$[M]\{\ddot{Y}\} + [C]\{\dot{Y}\} + [K]\{Y\} = \{f(t)\} \tag{6}$$

Where $[M]$, $[C]$ and $[K]$ are the $N \times N$ mass. Damping, and stiffness matrices of the structure repectively. Is the number of degree of freedom. $\{Y\}$, $\{\dot{Y}\}$ and $\{\ddot{Y}\}$ are the displacement,velocity and acceleration vector, $\{f(t)\}$ is the $N \times 1$ white noise random external excitation vector,which is loaded in the form of spectrum.It can be assumed that power spectral density matrix of the external excitation vector $\{f(t)\}$ can be expressed as:

$$S_f(\omega) = S \tag{7}$$

where S is $N \times N$ semi-positive real symmetric matrix, S would be a diagonal matrix if all degree of freedom are unrelated, i.e., $S_I = S_{II}, S_{ij} = 0 (i \neq j)$.

The corresponding undamped free vibration equation of (6) can be written as:

$$[M]\{\ddot{Y}\} + [K]\{Y\} = 0 \tag{8}$$

Assume that the solution for equation (8) is $\varphi sin \omega t$,then the characteristic vector equation for the equation (8) can be expressed as:

$$([K] - \omega_i^2 [M])\phi_i = 0 \tag{9}$$

In equation (9), ω_i and ϕ_i are the *i*-th circle natural frequency and mode shape.We consider that the mode shapes can be normalized with respect to the mass matrix and satisfy equation (10) under the proportional damping.

$$\begin{cases} \{\phi\}^T [M]\{\phi\} = I \\ \{\phi\}^T [K]\{\phi\} = diag(\omega_i^2) \\ \{\phi\}^T [C]\{\phi\} = diag(2\xi_i \omega_i) \end{cases} \tag{10}$$

Where $\{\phi\} = \{\phi_1, \phi_2, \cdots, \phi_i, \cdots \phi_N\}$ and ξ_i is the damping ratio of the *i*-th mode,assume that

$$\{Y\} = \{\phi\} \tag{11}$$

Equation(6) can be written as:

$$\ddot{y} + diag(2\xi_i \omega_i)\dot{y} + diag(\omega_i^2)y = \{\phi\}f(t) = F(t) \tag{12}$$

Then the correlation matrix of mode response y_r and y_s can be expressed as:

$$C_{rs} = \frac{\pi d_{rs} a_{rs} r_{rs}}{\eta_r \eta_s \omega_r \omega_s} \tag{13}$$

where

$$\eta_r = \sqrt{1 - \xi_r^2}, \qquad \eta_s = \sqrt{1 - \xi_s^2}, \qquad d_{rs} = \{\phi\}^T S \{\phi\} \tag{14a}$$

$$a_{rs} = \xi_r \omega_r + \xi_s \omega_s, \qquad b_{rs} = \eta_r \omega_r + \eta_s \omega_s, \qquad e_{rs} = \eta_r \omega_r - \eta_s \omega_s \tag{14b}$$

$$g_{rs} = \left(a_{rs}^2 + b_{rs}^2\right)^{-1}, \qquad Q_{rs} = \left(a_{rs}^2 + e_{rs}^2\right)^{-1}, \qquad \gamma_{rs} = -g_{rs} + Q \tag{14c}$$

3.Element topological variables and interpolation functions

A topological optimization for minimizing the structural weight while satisfying dynamic stress response constraints limits is built in this paper,In the topological optimization with dynamic response constraints,numerical examples show that the solution for dynamic response value and its sensitivity would be wrong because of interpolation schemes without enough accuracy,These interpolation schemes,such as Solid Isotropic Material with Penalty(SIMP),which cannot achieve the matching punishment between the stiffness and mass matrix,and lead to localized modes, reduce the accuracy of modal analysis.

Referring to the Rational Approximation for Material Properties(RAMP) and the ICM (Independent, Continuous and Mapping) method,the weigth.stiffness matrix ,stress and mass matrix of an element are recognized by following functions,respectively

$$w_i = f_w(\rho_i)w_i^0, \quad K = f_K(\rho_i)K_i^0, \quad [S]^k = f_s(\rho_i)[S]_i^0 \quad M^i = f_M(\rho_i)M_i^0 \tag{15a}$$

where

$$f_w(\rho_i) = \rho_I^{\alpha_w}, f_K(\rho_i) = \frac{\rho_i}{1 + \nu(1 - \rho_i)}, \quad f_s(\rho_i) = \rho_I^{\alpha_s}, \quad f_M(\rho_i) = \rho_I^{\alpha_M} \tag{15b}$$

where are the weight,stiffness matrix,stress and mass matrix of the *i*-th element,respectively,and $w_i^0, K_i^0, [S]_i^0$ and M_i^0 are orginal weight,orginal stiffness and mass matrix of the i-th element,respectively.In this work, $\alpha_w = \alpha_M = 1.25$, $\alpha_s = 5$ and $\nu = 6$ are used.Due to the rational function in the RAMP interpolation scheme, the derivative of the function isn't zero even when ρ_i tends to zero,which can prove the effect of weak material on stiffness,and prevent the localized modes.

4.Optimization problem statement

The topological optimization model for minimizing the structural weight with dynamic stress response constraints limits can be formulated as follows:

$$
\begin{cases}
\min : W \\
s.t.: \sigma_{vm,i}^2 \le (\sigma_{vm,i}^2)^U
\end{cases}
\tag{16}
$$

For the purpose of obtaining predominantly black-and-white design in this paper, the vary dynamic response constraints method is adopted,Then,equation(16) can be transferred into equation (17):

$$
\begin{cases}
\min : f = \sum_{i=1}^Q \rho_i^\alpha w_i^0 + \gamma \sum_{i=1}^Q (1-\rho_i^{\frac{\alpha}{2}})\rho_i^{\frac{\alpha}{2}} w_i^0 \\
\max_{i=1,2,\cdots Q} (\sigma_{vm,i}^2) \le (\sigma_{vm,i}^2)^{(k+1),U} \\
s.t.: \frac{1}{Q}\sum_{i=1}^Q (\sigma_{vm,i}^2) \le \frac{1}{Q}\sum_{i=1}^Q (\sigma_{vm,i}^2)^{(k+1),U}
\end{cases}
\tag{17}
$$

Where γ is a weighting experience parameter; is a varying constraint limit,and $(\sigma_{vm,i}^2)^{(k+1),U}$ is expressed as:

$$
(\sigma_{vm,i}^2)^{(k+1),U} = \begin{cases}
(\sigma_{vm,i}^2)^{(k)} + \min(\beta(\sigma_{vm,i}^2)^{(k)},(\alpha_L(\sigma_{vm,i}^2)^U - (\sigma_{vm,i}^2)^{(k)}).(\sigma_{vm,i}^2)^{(k)} \le (\sigma_{vm,i}^2)^U \\
(\sigma_{vm,i}^2)^{(k)} - \left| \min(\beta(\sigma_{vm,i}^2)^{(k)},(\alpha_L(\sigma_{vm,i}^2)^U - (\sigma_{vm,i}^2)^{(k)}) \right|.(\sigma_{vm,i}^2)^{(k)} \le (\sigma_{vm,i}^2)^U
\end{cases}
\tag{18}
$$

Where β is a dynamic response limit change factor, α_L is a relaxation coefficient for the dynamic response constraints, $(\sigma_{vm,i}^2)^{(k+1),U}$ are varied according to equation (18)at each outer loop iteriation step, $(\sigma_{vm,i}^2)^{(k)}$ is the dynamic stress response of the *i*-th element of the structure at the *k*-th loop iteration step.

5.Sensitivity analysis on the dynamic stress response

Assume that $x_i = \dfrac{1}{\rho_i}$,the sensitivity of the constraint function with respect to x_i can be expressed as:

$$
\frac{\partial(\sigma_{vm,k}^2)}{\partial x_i} = trace(A\{S\}^k \frac{\partial E[\{u\}^k \{u\}^{kT}]}{\partial x_i}\{S\}^{kT}) = trace(A\{S\}^k \frac{\partial C_{rs}}{\partial x_i}\{S\}^{kT})
\tag{19}
$$

$$\frac{\partial(C_{rs})}{\partial x_i} = -\frac{\pi d_{rs} a_{rs} r_{rs}}{\eta_r \eta_s \omega_r^2 \omega_s^2}\left[\frac{\omega_s}{2\omega_r}\frac{\partial(\omega_r^2)}{\partial x_i} + \frac{\omega_r}{2\omega_s}\frac{\partial(\omega_s^2)}{\partial x_i}\right] +$$

$$\frac{\pi}{\eta_r \eta_s \omega_r \omega_s}\left[\frac{\partial(d_{rs})}{\partial x_i}a_{rs}r_{rs} + d_{rs}\frac{\partial(a_{rs})}{\partial x_i}r_{rs} + d_{rs}a_{rs}\frac{\partial(r_{rs})}{\partial x_i}\right] \qquad (20)$$

$$\left\{\begin{array}{l}
\dfrac{\partial(d_{rs})}{\partial x_i} = \sum_{i=1}^{n}\left[\dfrac{\partial(\phi_{ir})}{\partial \eta_j}\varphi_{is} + \dfrac{\partial(\phi_{is})}{\partial \eta_j}\varphi_{ir}\right]S_i \\[4mm]
\dfrac{\partial(a_{rs})}{\partial x_i} = \dfrac{\xi_r}{2\omega_r}\dfrac{\partial(\omega_r^2)}{\partial x_i} + \dfrac{\xi_s}{2\omega_s}\dfrac{\partial(\omega_s^2)}{\partial x_i} \\[4mm]
\dfrac{\partial(b_{rs})}{\partial x_i} = \dfrac{\varsigma_r}{2\omega_r}\dfrac{\partial(\omega_r^2)}{\partial x_i} + \dfrac{\varsigma_s}{2\omega_s}\dfrac{\partial(\omega_s^2)}{\partial x_i} \\[4mm]
\dfrac{\partial(e_{rs})}{\partial x_i} = \dfrac{\varsigma_r}{2\omega_r}\dfrac{\partial(\omega_r^2)}{\partial x_i} - \dfrac{\varsigma_s}{2\omega_s}\dfrac{\partial(\omega_s^2)}{\partial x_i} \\[4mm]
\dfrac{\partial(r_{rs})}{\partial x_i} = g_{rs}^2\left[2a_{rs}\dfrac{\partial(a_{rs})}{\partial x_i} + 2b_{rs}\dfrac{\partial(b_{rs})}{\partial x_i}\right] - Q_{rs}^2\left[2a_{rs}\dfrac{\partial(a_{rs})}{\partial x_i} + 2e_{rs}\dfrac{\partial(e_{rs})}{\partial x_i}\right]
\end{array}\right. \qquad (21)$$

the sensitivities of the circle natural frequency and the mode shape with the respect to x_i can be respectively written as:

$$\frac{\partial \omega_k}{\partial x_i} = \frac{1}{2\omega_k}\phi_k^T\left(\frac{\partial[K]}{\partial x_i} - \omega_k^2\frac{\partial[M]}{\partial x_i}\right)\phi_k \qquad (22)$$

$$\frac{\partial \phi_k}{\partial x_i} = \sum_{p=1}^{N}b_{kp}^{ij}\phi_p \qquad b_{kp}^{ij} = \begin{cases}\dfrac{\phi_k^T\left(\dfrac{\partial[K]}{\partial x_i} - \omega_k^2\dfrac{\partial[M]}{\partial x_i}\right)\phi_p}{\omega_k^2 - \omega_p^2} & k \neq p \\[6mm] -\dfrac{1}{2}\phi_k^T\dfrac{\partial[M]}{\partial x_i}\phi_k & k = p\end{cases} \qquad (23)$$

$$\frac{\partial[K]}{\partial x_i} = -\frac{(1+\nu)}{((1+\nu)x_i - \nu)^2}[K]_i^0 \quad , \qquad \frac{\partial[M]}{\partial x_i} = -\frac{\alpha}{x_i^{\alpha+1}}[M]_i^0 \qquad (24)$$

In order to deal with checkboards problem of topological optimization,the approach proposed by Sigmund and Petersson is adopted to redistribute the sensitivity of the dynamic stress response. Then the dynamic stress constraint expressions in equation (17) can be approximated by their one-order approximation functions，and the objective function in equation (17) can be approximated by its second-order approximation function, and solving theapproximating quadratic programming problem of equation.(17) can be transferred into solving a dualprogramming problem by using the dual theory.

6.Numerical examples

A beam like 2D structure is shown in Fig.1. The beam with dimensions 0.3 m×0.1m is simply supported at both ends. The design domain 0.2 m×0.1 m is equally divided into 80×40 four-node plane stress elements. The material is assumed with Young's modulus E = 210GPa, Poisson's ratio v = 0.3, and mass density = 7800 kg/m3. A concentric force with auto-power spectral is applied at the middle point A along vertical directions at the bottom of the plate as shown in Fig. 1 and all mode damping ratios n = 0.02 are specified, the prescribed dynamic stress response constraint limit is 2.05×10^{17}.

Here, the parameter $\beta = 0.046$ and $\alpha_L = 1.02$ is selected,Fig. 2 is the optimal topology obtained by the proposed method. The optimization histories of the weight and the maximum dynamic stress response for the truncated mode number 16 are illustrated in Figs. 2 and 3, respectively. The optimaltopology obtained by the proposed method is shown in Fig. 4.
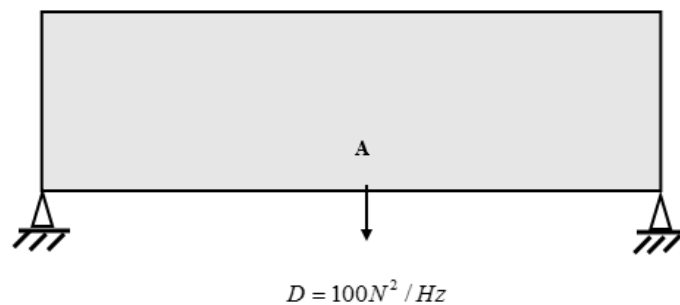

$$D = 100N^2 / Hz$$

Fig. 1. The initial structure model
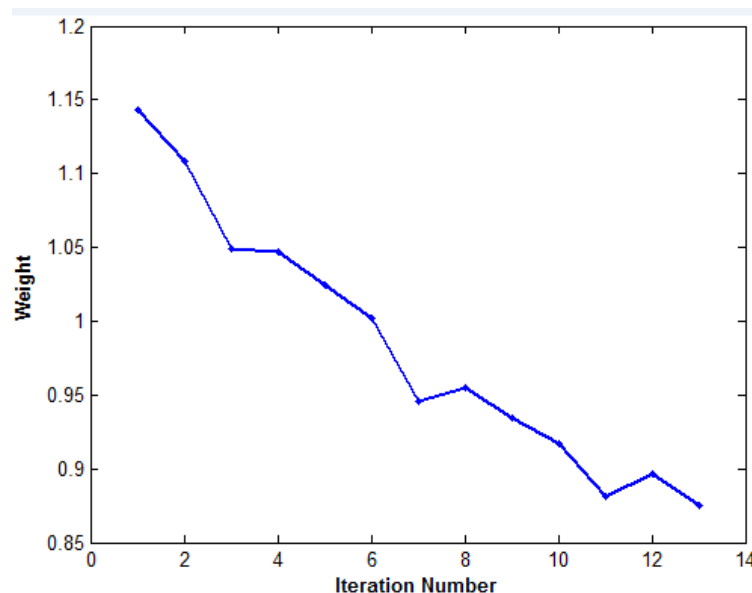

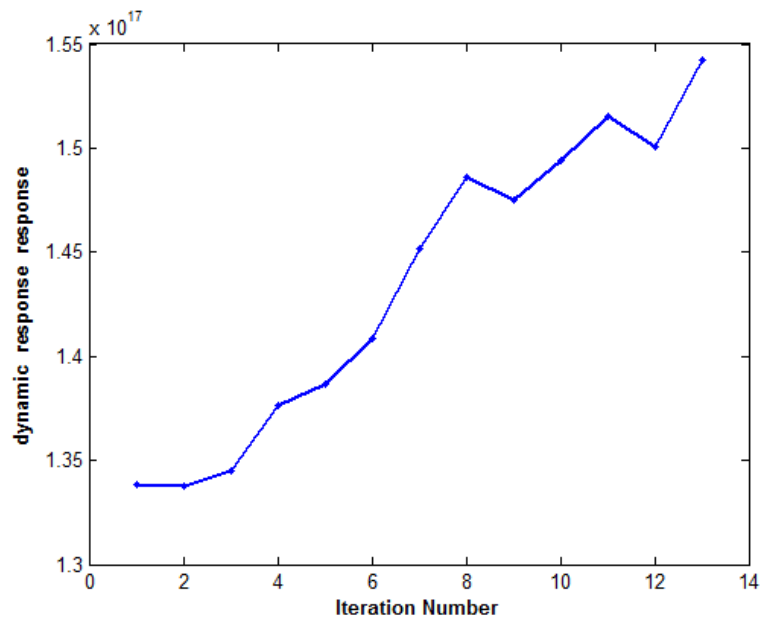**Fig. 2.** the optimization history of the topology weight

Fig. 3. the optimization history of the maximum dynamic stress response

Fig. 4. The optimal topology of the plane stress plate obtained by the proposed method

7.Conclusions

(1)A methodology for the topology optimization of continuum structures with with dynamic stress response constraints is proposed for minimizing the structural weight. Ramp interpolation schemes is used to prevent localized modes,and a vary dynamic response constraints method is adopted to make sure that the approximate expression is effective and avoid objective oscillation phenomenon in optimization iterations.The sensitivity of the dynamic stress response with respect to the design variable is derived.

(2)The example is used to prove the validity and efficiency of the proposed method for the dynamic stress response constraints optimization problem,the optimized topology can stably converge to optimal solutions. The obtained optimal design with prescribed dynamic stress response by the proposed method, will have a lower weight than the initial, guess design. The proposed continuum structural topological optimization with dynamic stress response constraints are effective, and undesired localized modes can be prevented.

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