

Dynamic Relaxation Method with Critical Damping for Nonlinear Analysis of Reinforced Concrete Elements

* Anna Szcześniak¹⁾ and Adam Stolarski²⁾

^{1), 2)} *Faculty of Civil Engineering and Geodesy, Military University of Technology,
Warsaw, Poland*

¹⁾ anna.szczesniak@wat.edu.pl

ABSTRACT

The method for analysis of the nonlinear behavior of reinforced concrete elements subjected to short-term static loads, called as the Dynamic Relaxation Method, is presented in the paper. In this method, the pseudo-dynamic problem can be reduced to analysis of a static problem by introduction the critical damping, and considering the process as an aperiodic motion that approaches the equilibrium state under the acting load. For the application of complete form of the method, considerations in the range of inelastic properties for the structural materials (Szcześniak 2016), as well as the modelling of deformation processes of reinforced concrete bar elements, are carried out. To increase the effectiveness of the method in the post-critical range of analysis, the path tracking for multiple variables method of solution is introduced to the solution of the system of dynamic nonlinear equilibrium equations. After including the arc-length parameter to the solution, the global softening of the structural element in post-critical range, which leads to realization of failure mechanism, can be analyzed. The results of numerical tests are carried out for reinforced concrete beam and eccentrically loaded column. Comparative analysis of the obtained results with experimental (Foley 1998, Lloyd 1996), numerical and analytical results presented in the literature demonstrated that the proposed computational method is very effective.

1. INTRODUCTION

The main purpose of this work was to develop a computational model of reinforced concrete bar elements, which can be used to study the behavior of an element going from purely elastic, to elastic-plastic, and then to stress softening and failure of the cross-section.

To obtain a solution, the modeling range of the inelastic material properties, the deformation processes of the bar structural elements, and the numerical solutions, must be considered. The properties of the structural material were modeled by

¹⁾ Adjunct

²⁾ Professor

assuming a reduced plane compressive/tensile stress state with shear. An elastic–plastic material model with hardening for the reinforcing steel was used. The elastic–plastic material model was developed by considering the stress softening and degradation of the deformation modulus for the concrete. Analysis of the structural element includes a description of the behavior of an eccentrically compressed reinforced concrete element, modeled as a bar system under static loading. The equations for moderately large displacements of the bar include the influence of the longitudinal deformations, changes to the rotation angle of the cross-section (curvature), and isochoric strains. These were used as the basis of theoretical model of the behavior of the structural element.

A load capacity analysis technique for a structural system was developed using the finite difference method, and the dynamic relaxation method (DRM) was applied to solve the nonlinear equilibrium equations. In this method, the problem can be reduced to analysis of a pseudo-dynamic process by considering critical damping as an aperiodic motion that approaches the equilibrium state under the acting load. The solution to this problem can be obtained using an iterative process with pseudo-time steps. The DRM is used to analyze highly nonlinear problems (Rodriguez et al. 2011, Hincz 2009). The characteristics of this method were presented by Underwood (1983). Rezaiee-Pajand and Alamatian (2010) proposed further modifications to improve the convergence and accuracy of the solutions.

The arc-length parameter on the equilibrium path was introduced to the proposed numerical procedure to increase the efficiency of the method. This is referred to as the arc-length method, and it simultaneously determines the displacements and load parameter by tracing the equilibrium path that contains the local singular points. Riks introduced the precursor of this method in Riks (1972). Subsequent developments and interpretations of the arc-length method were presented by Ramm (1981), Crisfield (1981 and 1996), Schweizerhof and Wriggers (1986), among others. A combination of the dynamic relaxation and arc-length methods was presented by Pasqualino (2004). This was then applied to practical analyses of structural members by Remesh and Krishnamoorthy (1993), Pasqualino and Estefen (2001).

In this paper, a general computational algorithm that is based on developed effort analysis methods for reinforced concrete elements was proposed. This technique was implemented in Fortran to calculate strains, stresses, internal forces, and displacements. To verify the accuracy of the proposed method, numerical studies of bar structural elements, were carried out. Obtained numerical results were compared with the results of experimental, analytical, and theoretical works from the existing literature.

2. MODELING OF STRUCTURAL MATERIALS

The elastic–plastic model with regard to material hardening was applied for the reinforcing steel. The reduced plane stress state (compression/tension with shear), because of the nature of the reinforcing slender bars in the reinforced concrete element, was considered.

$$\sigma_{11}^n = \frac{\sigma_{11}^{n-1} + E_s \Delta \varepsilon_{11}^n}{1 + \frac{2}{3} E_s \Delta A^n} \quad \text{and} \quad \sigma_{12}^n = \frac{\sigma_{12}^{n-1} + 2\mu_s \Delta \varepsilon_{12}^n}{1 + 2\mu_s \Delta A^n}, \quad (1)$$

where E_s is the deformation modulus of steel, $\mu_s = \frac{E_s}{2(1+\nu_s)}$ is the shear modulus,

ΔA^n is the scalar multiplier that defines the state of stress according to the associated plastic flow rule for the Huber-Mises-Hencky yield criterion, n is the step number of the instantaneous strain-stress state in subsequent time instant $t_n = t_{n-1} + \Delta t$, and Δt is the time step increment.

An elastic-plastic material model for concrete that considers material softening and the degradation of the deformation modulus was developed. A reduced plane stress state for the compression/tension range with shear ($\sigma_{11}, \sigma_{22} = 0, \sigma_{12}$) was assumed.

The concrete model describes the incremental equations for stresses while considering the limitations that are a result of the yield condition. That is,

$$\left\{ \begin{array}{l} \bar{\sigma}_{11}^n = \sigma_{11}^{n-1} + E_c \Delta \varepsilon_{11}^n \\ \bar{\sigma}_{12}^n = \sigma_{12}^{n-1} + 2\mu_c \Delta \varepsilon_{12}^n \end{array} \right. \quad \sigma_{11}^n = \left\{ \begin{array}{l} \bar{\sigma}_{11}^n \quad d l a \quad -f_t^n < \bar{\sigma}_{11}^n \leq f_c^n \\ -f_t^n \quad d l a \quad \bar{\sigma}_{11}^n < -f_t^n \\ f_c^n \quad d l a \quad \bar{\sigma}_{11}^n > f_c^n \end{array} \right. ,$$

$$\sigma_{12}^n = \left\{ \begin{array}{l} \bar{\sigma}_{12}^n \quad d l a \quad |\sigma_{12}^n| \leq f_s^n \\ f_s^n \quad d l a \quad |\sigma_{12}^n| > f_s^n \end{array} \right.$$

(2)

where n is the instantaneous step of the stress-strain state, $\varepsilon_{11}^n, \varepsilon_{12}^n, \Delta \varepsilon_{11}^n = \varepsilon_{11}^n - \varepsilon_{11}^{n-1}$, and $\Delta \varepsilon_{12}^n = \varepsilon_{12}^n - \varepsilon_{12}^{n-1}$ are the known strains and strain increments, f_{c0} , f_{t0} , and f_{s0} are the initial compressive, tensile, and shear strengths, E_{c0} is the initial modulus of

deformation, $\mu_c = \frac{E_c}{2(1+\nu_{c0})}$ is the shear deformation modulus, ν_{c0} is the initial

Poisson's ratio, $\varepsilon_{c0} = \frac{f_{c0}}{E_{c0}}$ and $\varepsilon_{t0} = \frac{f_{t0}}{E_{c0}}$ are the elastic limit strains in compression

and tension, ε_{fc} and ε_{uc} are the strain limits for the perfectly plastic flow and the material softening range in compression, and ε_{ut} is the strain limit for the material softening range in tension.

The yield criterion used in concrete model is consistent with experimental results for the reduced plane stress state. This is also confirmed by comparing proposed model with that described by Stolarski (1991), which was calibrated with experimental results.

3. FUNDAMENTAL EQUATIONS

3.1 Equations of motion

The equations of dynamic motion for a reinforced concrete bar element characterized by the unit mass (μ), the unit mass of inertia (j), and the unit damping coefficients for the displacements (c) and rotations (c_f), were derived. Problem of the damping coefficient estimation was detailed described in the paper of Szcześniak & Stolarski (2015).

The differential equilibrium equations were defined in the global Cartesian coordinate system ($\{x(u), z(w)\}$), as shown in Fig. 4. They have the form

$$\begin{cases} -\frac{\partial(N \cos \theta)}{\partial s} + \frac{\partial(Q \sin \theta)}{\partial s} + p_x(s) - \mu \ddot{u} - c \dot{u} = 0 \\ -\frac{\partial(N \sin \theta)}{\partial s} - \frac{\partial(Q \cos \theta)}{\partial s} - p_z(s) + \mu \ddot{w} + c \dot{w} = 0 \\ \frac{\partial M}{\partial s} - Q - j \ddot{\theta} - c_f \dot{\theta} = 0 \end{cases}$$

(3)

where N is the internal longitudinal force, Q is the transversal force, M is the bending moment, $\{p_x, p_y\}$ are the external loads, $\{\mu \ddot{u}, \mu \ddot{w}, j \ddot{\theta}\}$ are the inertial longitudinal, transverse, and rotational forces, ds is the length of the deformed element, and θ is the slope angle.

3.2 Equations of internal equilibrium in the cross-section

The bar's cross-section was discretized to produce a computational model. The cross-section of the concrete was divided into layers that were Δh - thick, and, within this, the areas of the two steel layers, were defined A_{s1} and A_{s2} .

The strain states in the different cross-sectional layers, for a given time step $t_n = t_{n-1} + \Delta t$, are defined as

$$\begin{cases} \varepsilon_{11r}^n = e^n + z_r \cdot \kappa^n, & z_r = (z_k, z_{s1}, z_{s2}), \quad k = 1, 2, \dots, K \\ \varepsilon_{12r}^n = \frac{1}{2} \gamma^n \end{cases},$$

(4)

If the longitudinal strain of the central axis (e^n), the change in the average rotation angle of the cross-section (κ^n), and the average angle of the non-dilatational strain

(γ^n) were known, then, after using the material models, the longitudinal force (N^n),

bending moment (M^n), and transverse force (Q^n) can be determined from the equilibrium equations of the cross-section. That is,

$$\begin{cases} N^n = \sum_{k=1}^K \sigma_{11,k}^n \cdot A_{c,k} + \sigma_{11,s1}^n \cdot A_{s1} + \sigma_{11,s2}^n \cdot A_{s2} \\ M^n = \sum_{k=1}^K \sigma_{11,k}^n \cdot A_{c,k} \cdot z_k + \sigma_{11,s1}^n \cdot A_{s1} \cdot z_{s1} + \sigma_{11,s2}^n \cdot A_{s2} \cdot z_{s2} \\ Q^n = \sum_{k=1}^K \sigma_{12,k}^n \cdot A_{c,k} + \sigma_{12,s1}^n \cdot A_{s1} + \sigma_{12,s2}^n \cdot A_{s2} \end{cases},$$

(5)

where $A_{c,k}$ is the cross-section area of the concrete layer, and $A_{s1,2}$ is the cross-section area of the tensile/compressive reinforcing steel.

4. SOLUTION OF THE SYSTEM OF EQUILIBRIUM EQUATIONS

4.1 Numerical solution of equations of motion

The system of equations in (5) was solved using a numerical method and a discretization with respect to time. For this purpose, the direct differential method with respect to time, was applied.

In this method, acceleration $\ddot{\mathbf{q}} = (\ddot{u}_i, \ddot{w}_i, \ddot{\phi}_{i1})^T$ and displacement velocity $\dot{\mathbf{q}} = (\dot{u}_i, \dot{w}_i, \dot{\phi}_{i1})^T$ were approximated in the system of equations in (3) at time instants $t_{n-1} = t_n - \Delta t$, $t_n = n\Delta t$, and $t_{n+1} = t_n + \Delta t$ using

$$\ddot{\mathbf{q}}^n = \frac{\dot{\mathbf{q}}^n - \dot{\mathbf{q}}^{n-1}}{\Delta t} \quad \text{and} \quad \dot{\mathbf{q}}^n = \frac{\Delta \mathbf{q}^n}{\Delta t},$$

(6)

Using these approximations, the system of equations in (3) becomes

$$\mathbf{G}(\mathbf{q}, \lambda) = \Delta \mathbf{q}^n - \Delta \mathbf{q}_G^n - \Delta \lambda^n \Delta \mathbf{q}_P^n = 0,$$

(7)

where $\Delta \mathbf{q}^n = (\Delta u_i^n, \Delta w_i^n, \Delta \phi_{i1}^n)^T$ is a vector of searched displacement increments,

$$\Delta \mathbf{q}_P^n = \begin{cases} a_i P_x(s_i) \\ a_i P_z(s_i) \\ a_{fi} [M_{i_{b1}} k(x_i - x_{i_{b1}}) + M_{i_{b2}} k(x_i - x_{i_{b2}})] \end{cases} \quad \text{is a vector of components of the}$$

displacement increments for the total load, $k(f) = \begin{cases} 1 & \text{if } f = 0 \\ 0 & \text{if } f \neq 0 \end{cases}$ is a selection operator,

i is the node number, and $i1 = i + 1$, $i0 = i - 1$ are the segment numbers of the internal spatial division,

$$\Delta \mathbf{q}_G^n = \left\{ \begin{array}{l} b_i \Delta u_i^{n-1} + a_i [-N_{i1} \cos \theta_{i1} + N_{i0} \cos \theta_{i0} + Q_{i1} \sin \theta_{i1} - Q_{i0} \sin \theta_{i0}] \\ b_i \Delta w_i^{n-1} + a_i [N_{i1} \sin \theta_{i1} - N_{i0} \sin \theta_{i0} + Q_{i1} \cos \theta_{i1} - Q_{i0} \cos \theta_{i0}] \\ b_{fi} \Delta \varphi_{i1}^n + a_{fi} (M_{i+2} - M_i - Q_{i1} \Delta s_{i1}) \end{array} \right\} + \lambda^{n-1} \Delta \mathbf{q}_P^n \text{ is a vector}$$

of components of the displacement increments for the load parameter λ^{n-1} from the previous time step, and $b_i = \frac{m_i}{m_i + C_i \Delta t}$, $a_i = \frac{\Delta t^2}{m_i + C_i \Delta t}$, $b_{fi} = \frac{j_i}{j_i + C_{fi} \Delta t}$ and

$$a_{fi} = \frac{\Delta t^2}{j_i + C_{fi} \Delta t}, C_i - \text{the damping factor for displacements in main node, } C_{fi} - \text{the}$$

damping factor for the rotations of the segment division.

The initial conditions for time step $t_{n=0} = 0$ have the form

$$\Delta \mathbf{q}^{n=-1} = 0,$$

(8)

The iterative procedure terminates when the solutions of displacements in subsequent time instants have converged, i.e.,

$$\|\Delta \mathbf{q}^n\| \leq \varepsilon_{\Delta q},$$

(9)

The time step Δt is determined so that the numerical integration is stable, and has the form

$$\Delta t = \alpha_r \min \{ \Delta t_{(l)}, \Delta t_{(b)}, \Delta t_{(f)} \},$$

(10)

where $\Delta t_{(l)} = \Delta \bar{s}_i \min \sqrt{\frac{\rho_{cs}}{E_{c0} + \rho_{s \max} E_s}}$ is the critical time step for the longitudinal elastic

vibration problem, $\Delta t_{(b)} = \frac{1}{2} \Delta \bar{s}_i^2 \min \sqrt{\frac{\mu(s_i)}{B_{cs}}}$ is the critical time step for the elastic

bending problem, $\Delta t_{(f)} = \Delta \bar{s}_i \min \sqrt{\frac{\rho_{cs}}{\mu_c}}$ is the critical time step for the elastic isochoric

wave propagation, $\Delta \bar{s}_i = \frac{\Delta s_{i1} + \Delta s_{i0}}{2}$, $\alpha_r = \langle 0.2, 0.9 \rangle$ is a safety factor for the time

step, $\mu(s_i) = \rho_{cs} A_{cs}$ is the unit mass of the element, ρ_{cs} is the density of the reinforced concrete, $B_{cs} = E_{c0} J_{cs}$ is the bending rigidity of the reinforced concrete cross-section in the elastic range, J_{cs} is the moment of inertia of the uncracked (elastic) reinforced concrete cross-section, and $\rho_{s \max}$ is the largest total reinforcement ratio for the entire reinforced concrete element.

4.2 Solution to the constraints equation

The constraint equation has the form (Wriggers 2008)

$$f(\mathbf{q}, \lambda) = \Delta \mathbf{q}_m^T \Delta \mathbf{q}_m + \Delta \lambda_m^2 - \Delta l^2 = 0, \quad (11)$$

where Δl is the increment (parameter) of the arc length on the solution path, $\mathbf{q} = (u, w, \varphi)^T$ is a vector of the searched displacements, $\Delta \lambda_m = \lambda^n - \lambda_m$ is the increment of the current load parameter λ^n , and $\Delta \mathbf{q}_m = \mathbf{q}^n - \mathbf{q}_m$ is the increment of the current vector of unknown displacements \mathbf{q}^n in relation to the last convergent values of the load parameter λ_m and displacement vector $\Delta \mathbf{q}_m$, obtained with the assumed accuracy in the previous load step m .

The load parameter λ^n was determined by directly solving the constraints in Equation (11). The sign of the differentiator $g(\mathbf{q}) = \Delta l^2 - \Delta \mathbf{q}_m^T \Delta \mathbf{q}_m$, was checked, and the solutions were determined according to

$$\lambda_{1,2}^n = \lambda_m \pm \sqrt{\Delta l^2 - \Delta \mathbf{q}_m^T \Delta \mathbf{q}_m}, \text{ if } g(\mathbf{q}) \geq 0$$

and

$$\lambda_{1,2}^n = \lambda_m \pm \sqrt{\Delta \mathbf{q}_m^T \Delta \mathbf{q}_m - \Delta l^2}, \text{ if } g(\mathbf{q}) < 0$$

Then, the solution that is the smallest "distance" away from the solution obtained in load step $m-1$, was selected. That is, the smallest angle (or the largest cosine of the angle) between the solutions in steps m and $m-1$,

$$g = \min(\vartheta_1, \vartheta_2) = \min\left(\arccos \frac{\Delta \mathbf{q}_{m,1} \Delta \mathbf{q}_{m-1}}{\Delta l^2}, \arccos \frac{\Delta \mathbf{q}_{m,2} \Delta \mathbf{q}_{m-1}}{\Delta l^2}\right), \quad (12)$$

Analyzing the conditions of the solution to the constraints in Equation (11), it follows that this criterion satisfies

$$\lambda^n = \lambda_m + \text{sign}(\Delta l^2 - \Delta \mathbf{q}_m^T \Delta \mathbf{q}_m) \sqrt{\text{abs}(\Delta l^2 - \Delta \mathbf{q}_m^T \Delta \mathbf{q}_m)}, \quad (13)$$

4.3 Conceptual algorithm for solving the extended system of equations

The constraints in Equation (11) together with the system of equations of motion in (7) form an extended system of equations. This system for any time instant $t_n = n\Delta t$ can be iteratively solved. This extended system of equations can be used to determine the searched displacement vector (\mathbf{q}) and the load parameter (λ) for the nonlinear equilibrium path containing the local limit points. This extended system was solved using a two-stage procedure. First, in the prediction stage, the load parameter (which depends on the displacement values from the previous iterative step) was determined using the constraints in (11). Second, in the correction stage, the new displacement

values in the next iterative step were determined using the system of motion equations in (7).

Fig. 1 shows the iterative scheme for solving the extended system of equations.

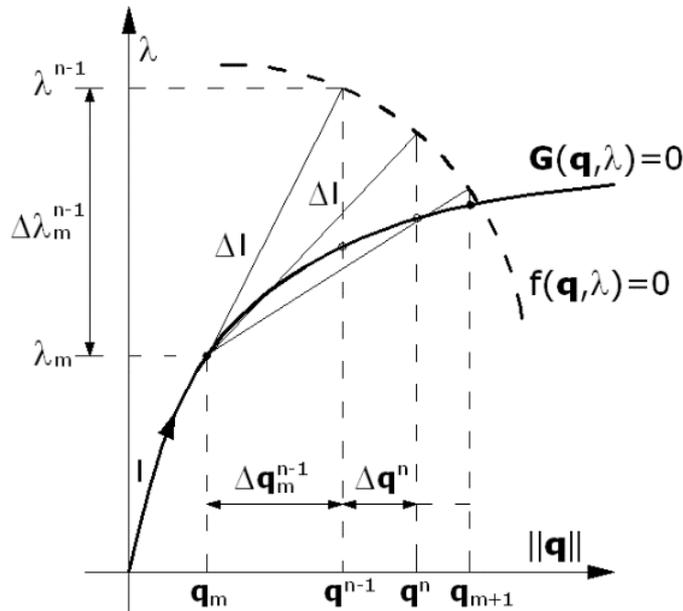


Fig. 1 Iterative scheme for solving the system of extended equations

The solution after n iterations is

$$\lambda^n = \lambda^{n-1} + \Delta\lambda^n, \quad \mathbf{q}^n = \mathbf{q}^{n-1} + \Delta\mathbf{q}^n, \quad (14)$$

The iterations are terminated after reaching a predefined accuracy (ε_G), that is,

$$\|\mathbf{G}(\mathbf{q}^n, \lambda^n)\| \leq \varepsilon_G, \quad (15)$$

On the basis of carried out numerical experiments, it was determined that the accuracy should satisfy $\varepsilon_G \in (0.1^{-3}, 0.1^{-6})$.

This method allows us to analysis of the static behavior of a physically and geometrically nonlinear structural element, which was described by system of dynamic equilibrium equations after applying the critical damping.

Figures 2 and 3 schematically show load and displacement in pseudo time function for the dynamic relaxation method before and after applying critical damping. In the elastic range using the critical damping allows determining the displacement values that correspond to the equilibrium state of the undamped vibrations. Whereas in inelastic range, the displacements determined after introducing critical damping are usually lower than displacement values corresponding to the equilibrium state of the undamped vibration.

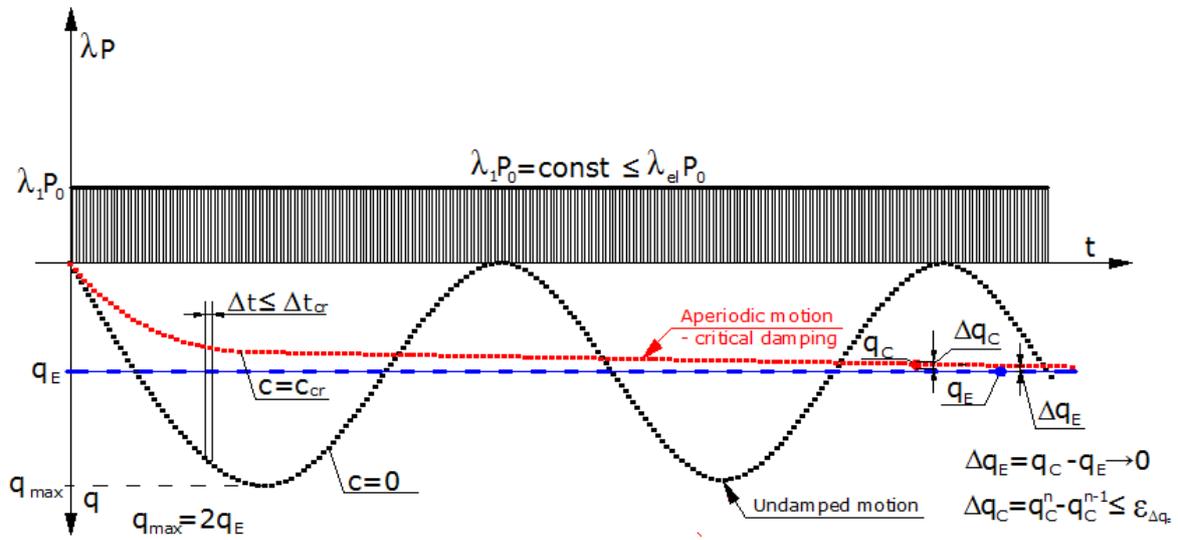


Fig. 2 Load and displacement vs. pseudo time for the dynamic relaxation method with critical damping in elastic range

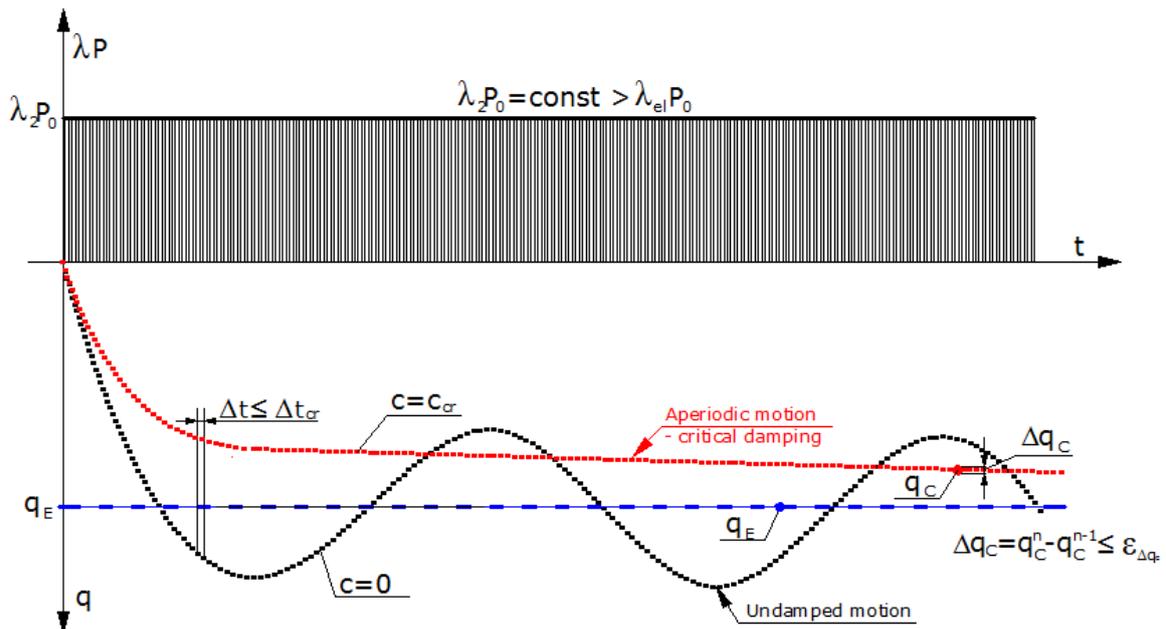


Fig. 3 Load and displacement vs. pseudo time for the dynamic relaxation method with critical damping in inelastic range

5. CONCLUSIONS

The paper presents a behavior analysis method for the bending and compression of reinforced concrete elements subjected to short-term static loads. The structural system analysis technique was developed using assumptions from the finite difference method. The central axis of the structural element was discretized, and the equilibrium equations and geometrical relationships in differential form were written. A joint-action rule was determined for the active concrete and steel layers of the discretized cross-sectional model.

A dynamic process that can describe the static problem by introducing critical damping, was analyzed. When this procedure is used to solve a system of nonlinear equilibrium equations it is referred to as the DRM. Using DRM, the equations of motion can be recursively solved in subsequent pseudo-time instants, for each node of the spatial division. There was no need to solve the system of algebraic equilibrium equations. The DRM was improved by introducing an additional constraint equation to the system of equilibrium equations, according to the assumptions of the arc length method. This method is called the DRM+AL, and can be used to solve the nonlinear equilibrium equations in the post-critical range.

When modeling the behavior of structural materials, the reduced plane compressive/tensile states with shear stress was considered. Equations derived from the theory of moderately large displacements in a bar system form the basis of structural behavior description, assuming infinitesimal deformations. This assumption is appropriate when describing strain and stress states in the concrete and steel layers of commonly used reinforced concrete elements.

The verification of developed method, numerical tests on a reinforced concrete beam and eccentrically loaded column, were carried out and indicate to high effectiveness of the method for a post-critical analysis.

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