$$\sigma_{11}^{n} = \frac{\sigma_{11}^{n-1} + E_s \varDelta\varepsilon_{11}^{n}}{1 + \frac{2}{3} E_s \varDelta\varLambda^{n}} \quad \text{and} \quad \sigma_{12}^{n} = \frac{\sigma_{12}^{n-1} + 2\mu_s \varDelta\varepsilon_{12}^{n}}{1 + 2\mu_s \varDelta\varLambda^{n}} \;,$$

(1)

where E_s is the deformation modulus of steel, $\mu_s = \frac{E_s}{2(1+\nu_s)}$ is the shear modulus, $\varDelta\varLambda^n$ is the scalar multiplier that defines the state of stress according to the associated plastic flow rule for the Huber-Mises-Hencky yield criterion, n is the step number of the instantaneous strain-stress state in subsequent time instant $t_n = t_{n-1} + \varDelta t$, and $\varDelta t$ is the time step increment.

An elastic–plastic material model for concrete that considers material softening and the degradation of the deformation modulus was developed. A reduced plane stress state for the compression/tension range with shear $(\sigma_{11}, \sigma_{22} = 0, \sigma_{12})$ was assumed.

The concrete model describes the incremental equations for stresses while considering the limitations that are a result of the yield condition. That is,

$$\begin{cases} \bar{\sigma}_{11}^{n} = \sigma_{11}^{n-1} + E_c \varDelta\varepsilon_{11}^{n} \quad \sigma_{11}^{n} = \begin{cases} \bar{\sigma}_{11}^{n} & d\,l\,a \quad -f_t^{\,n} < \bar{\sigma}_{11}^{n} \le f_c^{\,n} \\ -f_t^{n} & d\,l\,a \quad \bar{\sigma}_{11}^{n} < -f_t^{n} \\ f_c^{n} & d\,l\,a \quad \bar{\sigma}_{11}^{n} > f_c^{n} \end{cases} \\ \bar{\sigma}_{12}^{n} = \sigma_{12}^{n-1} + 2\mu_c \varDelta\varepsilon_{12}^{n} \quad \sigma_{12}^{n} = \begin{cases} \bar{\sigma}_{12}^{n} & d\,l\,a \quad |\sigma_{12}^{n}| \le f_s^{n} \\ f_s^{n} & d\,l\,a \quad |\sigma_{12}^{n}| > f_s^{n} \end{cases} \end{cases}$$

(2)

where n is the instantaneous step of the stress-strain state, $\varepsilon_{11}^{n}, \varepsilon_{12}^{n}, \varDelta\varepsilon_{11}^{n} = \varepsilon_{11}^{n} - \varepsilon_{11}^{n-1}$, and $\varDelta\varepsilon_{12}^{n} = \varepsilon_{12}^{n} - \varepsilon_{12}^{n-1}$ are the known strains and strain increments, f_{c0}, f_{t0}, and f_{s0} are the initial compressive, tensile, and shear strengths, E_{c0} is the initial modulus of deformation, $\mu_c = \frac{E_c}{2(1+\nu_{c0})}$ is the shear deformation modulus, ν_{c0} is the initial Poisson's ratio, $\varepsilon_{c0} = \frac{f_{c0}}{E_{c0}}$ and $\varepsilon_{t0} = \frac{f_{t0}}{E_{c0}}$ are the elastic limit strains in compression and tension, ε_{fc} and ε_{uc} are the strain limits for the perfectly plastic flow and the material softening range in compression, and ε_{ut} is the strain limit for the material softening range in tension.

The yield criterion used in concrete model is consistent with experimental results for the reduced plane stress state. This is also confirmed by comparing proposed model with that described by Stolarski (1991), which was calibrated with experimental results.

3. FUNDAMENTAL EQUATIONS

3.1 Equations of motion

The equations of dynamic motion for a reinforced concrete bar element characterized by the unit mass (μ), the unit mass of inertia (j), and the unit damping coefficients for the displacements (c) and rotations (c_f), were derived. Problem of the damping coefficient estimation was detailed described in the paper of Szcześniak & Stolarski (2015).

The differential equilibrium equations were defined in the global Cartesian coordinate system ($\{x(u), z(w)\}$), as shown in Fig. 4. They have the form

$$\begin{cases} -\dfrac{\partial(N\cos\vartheta)}{\partial s}+\dfrac{\partial(Q\sin\vartheta)}{\partial s}+p_x(s)-\mu\ddot{u}-c\dot{u}=0 \\ -\dfrac{\partial(N\sin\vartheta)}{\partial s}-\dfrac{\partial(Q\cos\vartheta)}{\partial s}-p_z(s)+\mu\ddot{w}+c\dot{w}=0 \\ \dfrac{\partial M}{\partial s}-Q-j\ddot{\varphi}-c_f\dot{\varphi}=0 \end{cases}$$

(3)

where N is the internal longitudinal force, Q is the transversal force, M is the bending moment, $\{p_x, p_y\}$ are the external loads, $\{\mu\ddot{u}, \mu\ddot{w}, j\ddot{\varphi}\}$ are the inertial longitudinal, transverse, and rotational forces, ds is the length of the deformed element, and θ is the slope angle.

3.2 Equations of internal equilibrium in the cross-section

The bar's cross-section was discretized to produce a computational model. The cross-section of the concrete was divided into layers that were Δh - thick, and, within this, the areas of the two steel layers, were defined A_{s1} and A_{s2}.

The strain states in the different cross-sectional layers, for a given time step $t_n = t_{n-1} + \Delta t$, are defined as

$$\begin{cases} \varepsilon_{11r}^{n} = e^{n} + z_r \cdot \kappa^{n}, \quad z_r = (z_k, z_{s1}, z_{s2}), \quad k = 1,2,...,K \\ \varepsilon_{12r}^{n} = \dfrac{1}{2}\gamma^{n} \end{cases},$$

(4)

If the longitudinal strain of the central axis (e^n), the change in the average rotation angle of the cross-section (κ^n), and the average angle of the non-dilatational strain

(γ^n) were known, then, after using the material models, the longitudinal force (N^n), bending moment (M^n), and transverse force (Q^n) can be determined from the equilibrium equations of the cross-section. That is,

$$\begin{cases} N^n = \sum_{k=1}^{K} \sigma_{11,k}^n \cdot A_{c,k} + \sigma_{11,s1}^n \cdot A_{s1} + \sigma_{11,s2}^n \cdot A_{s2} \\ M^n = \sum_{k=1}^{K} \sigma_{11k}^n \cdot A_{c,k} \cdot z_k + \sigma_{11,s1}^n \cdot A_{s1} \cdot z_{s1} + \sigma_{11,s2}^n \cdot A_{s2} \cdot z_{s2} \\ Q^n = \sum_{k=1}^{K} \sigma_{12,k}^n \cdot A_{c,k} + \sigma_{12,s1}^n \cdot A_{s1} + \sigma_{12,s2}^n \cdot A_{s2} \end{cases},$$

(5)

where $A_{c,k}$ is the cross-section area of the concrete layer, and $A_{s1,2}$ is the cross-section area of the tensile/compressive reinforcing steel.

4. SOLUTION OF THE SYSTEM OF EQUILIBRIUM EQUATIONS

4.1 Numerical solution of equations of motion

The system of equations in (5) was solved using a numerical method and a discretization with respect to time. For this purpose, the direct differential method with respect to time**,** was applied.

In this method, acceleration $\ddot{\mathbf{q}} = (\ddot{u}_i, \ddot{w}_i, \ddot{\varphi}_{i1})^T$ and displacement velocity $\dot{\mathbf{q}} = (\dot{u}_i, \dot{w}_i, \dot{\varphi}_{i1})^T$ were approximated in the system of equations in (3) at time instants $t_{n-1} = t_n - \Delta t$, $t_n = n\Delta t$, and $t_{n+1} = t_n + \Delta t$ using

$$\ddot{\mathbf{q}}^n = \frac{\dot{\mathbf{q}}^n - \dot{\mathbf{q}}^{n-1}}{\Delta t} \quad \text{and} \quad \dot{\mathbf{q}}^n = \frac{\Delta \mathbf{q}^n}{\Delta t},$$

(6)

Using these approximations, the system of equations in (3) becomes

$$\mathbf{G}(\mathbf{q}, \lambda) = \Delta \mathbf{q}^n - \Delta \mathbf{q}_G^n - \Delta \lambda^n \Delta \mathbf{q}_P^n = 0 ,$$

(7)

where $\Delta \mathbf{q}^n = \left(\Delta u_i^n, \Delta w_i^n, \Delta \varphi_{i1}^n\right)^T$ is a vector of searched displacement increments, $\Delta \mathbf{q}_P^n = \begin{cases} a_i P_x(s_i) \\ a_i P_z(s_i) \\ a_{fi}[M_{i_{b1}} k(x_i - x_{i_{b1}}) + M_{i_{bI}} k(x_i - x_{i_{bI}})] \end{cases}$ is a vector of components of the displacement increments for the total load, $k(f) = \begin{cases} 1 & if \quad f = 0 \\ 0 & if \quad f \neq 0 \end{cases}$ is a selection operator,

i is the node number, and $i1 = i+1$, $i0 = i-1$ are the segment numbers of the internal spatial division,

$$\Delta\mathbf{q}_G^n = \begin{Bmatrix} b_i \Delta u_i^{n-1} + a_i[-N_{i1}\cos\theta_{i1} + N_{i0}\cos\theta_{i0} + Q_{i1}\sin\theta_{i1} - Q_{i0}\sin\theta_{i0}] \\ b_i \Delta w_i^{n-1} + a_i[N_{i1}\sin\theta_{i1} - N_{i0}\sin\theta_{i0} + Q_{i1}\cos\theta_{i1} - Q_{i0}\cos\theta_{i0}] \\ b_{fi}\Delta\varphi_{i1}^n + a_{fi}(M_{i+2} - M_i - Q_{i1}\Delta s_{i1}) \end{Bmatrix} + \lambda^{n-1}\Delta\mathbf{q}_P^n$$ is a vector of components of the displacement increments for the load parameter λ^{n-1} from the previous time step, and $b_i = \frac{m_i}{m_i + C_i \Delta t}$, $a_i = \frac{\Delta t^2}{m_i + C_i \Delta t}$, $b_{fi} = \frac{j_i}{j_i + C_{fi}\Delta t}$ and $a_{fi} = \frac{\Delta t^2}{j_i + C_{fi}\Delta t}$, C_i - the damping factor for displacements in main node, C_{fi} - the damping factor for the rotations of the segment division.

The initial conditions for time step $t_{n=0} = 0$ have the form

$$\Delta\mathbf{q}^{n=-1} = 0 \quad , \tag{8}$$

The iterative procedure terminates when the solutions of displacements in subsequent time instants have converged, i.e.,

$$\left\| \Delta\mathbf{q}^n \right\| \le \varepsilon_{\Delta q} \quad , \tag{9}$$

The time step Δt is determined so that the numerical integration is stable, and has the form

$$\Delta t = \alpha_r \min\{\Delta t_{(l)}, \Delta t_{(b)}, \Delta t_{(f)}\} \quad , \tag{10}$$

where $\Delta t_{(l)} = \Delta \bar{s}_{i\,\min}\sqrt{\frac{\rho_{cs}}{E_{c0} + \rho_{s\,\max}E_s}}$ is the critical time step for the longitudinal elastic vibration problem, $\Delta t_{(b)} = \frac{1}{2}\Delta \bar{s}_{i\,\min}^2\sqrt{\frac{\mu(s_i)}{B_{cs}}}$ is the critical time step for the elastic bending problem, $\Delta t_{(f)} = \Delta \bar{s}_{i\,\min}\sqrt{\frac{\rho_{cs}}{\mu_c}}$ is the critical time step for the elastic isochoric wave propagation, $\Delta \bar{s}_i = \frac{\Delta s_{i1} + \Delta s_{i0}}{2}$, $\alpha_r = <0.2, 0.9>$ is a safety factor for the time step, $\mu(s_i) = \rho_{cs}A_{cs}$ is the unit mass of the element, ρ_{cs} is the density of the reinforced concrete, $B_{cs} = E_{c0}J_{cs}$ is the bending rigidity of the reinforced concrete cross-section in the elastic range, J_{cs} is the moment of inertia of the uncracked (elastic) reinforced concrete cross-section, and $\rho_{s\,\max}$ is the largest total reinforcement ratio for the entire reinforced concrete element.

4.2 Solution to the constraints equation

The constraint equation has the form (Wriggers 2008)

$$f(\mathbf{q},\lambda) = \Delta\mathbf{q}_m^T \Delta\mathbf{q}_m + \Delta\lambda_m^2 - \Delta l^2 = 0 \quad , \tag{11}$$

where Δl is the increment (parameter) of the arc length on the solution path, $\mathbf{q} = (u, w, \varphi)^T$ is a vector of the searched displacements, $\Delta\lambda_m = \lambda^n - \lambda_m$ is the increment of the current load parameter λ^n, and $\Delta\mathbf{q}_m = \mathbf{q}^n - \mathbf{q}_m$ is the increment of the current vector of unknown displacements $\mathbf{q}^n$ in relation to the last convergent values of the load parameter λ_m and displacement vector $\Delta\mathbf{q}_m$, obtained with the assumed accuracy in the previous load step m.

The load parameter λ^n was determined by directly solving the constraints in Equation (11). The sign of the differentiator $g(\mathbf{q}) = \Delta l^2 - \Delta\mathbf{q}_m^T \Delta\mathbf{q}_m$, was checked, and the solutions were determined according to

$\lambda_{1,2}^n = \lambda_m \pm \sqrt{\Delta l^2 - \Delta\mathbf{q}_m^T \Delta\mathbf{q}_m}$, if $g(\mathbf{q}) \geq 0$

and

$\lambda_{1,2}^n = \lambda_m \pm \sqrt{\Delta\mathbf{q}_m^T \Delta\mathbf{q}_m - \Delta l^2}$, if $g(\mathbf{q}) < 0$

Then, the solution that is the smallest "distance" away from the solution obtained in load step $m-1$, was selected. That is, the smallest angle (or the largest cosine of the angle) between the solutions in steps m and $m-1$,

$$\vartheta = \min(\vartheta_1, \vartheta_2) = \min\left(\arccos\frac{\Delta\mathbf{q}_{m,1}\Delta\mathbf{q}_{m-1}}{\Delta l^2}, \arccos\frac{\Delta\mathbf{q}_{m,2}\Delta\mathbf{q}_{m-1}}{\Delta l^2}\right) , \tag{12}$$

Analyzing the conditions of the solution to the constraints in Equation (11), it follows that this criterion satisfies

$$\lambda^n = \lambda_m + sign(\Delta l^2 - \Delta\mathbf{q}_m^T \Delta\mathbf{q}_m)\sqrt{abs(\Delta l^2 - \Delta\mathbf{q}_m^T \Delta\mathbf{q}_m)} \quad , \tag{13}$$

4.3 Conceptual algorithm for solving the extended system of equations

The constraints in Equation (11) together with the system of equations of motion in (7) form an extended system of equations. This system for any time instant $t_n = n\Delta t$ can be iteratively solved. This extended system of equations can be used to determine the searched displacement vector ($\mathbf{q}$) and the load parameter (λ) for the nonlinear equilibrium path containing the local limit points. This extended system was solved using a two-stage procedure. First, in the prediction stage, the load parameter (which depends on the displacement values from the previous iterative step) was determined using the constraints in (11). Second, in the correction stage, the new displacement values in the next iterative step were determined using the system of motion equations in (7).

Fig. 1 shows the iterative scheme for solving the extended system of equations.

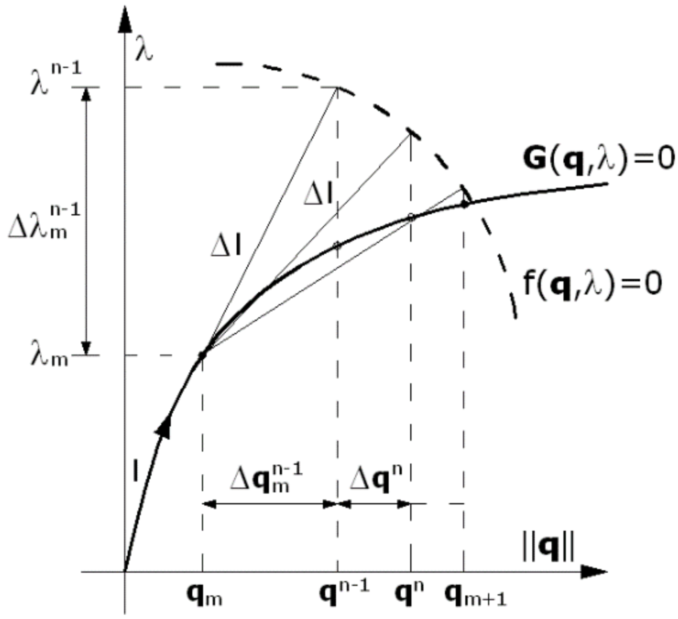

Fig. 1 Iterative scheme for solving the system of extended equations

The solution after n iterations is

$$\lambda^n = \lambda^{n-1} + \Delta\lambda^n \ , \quad \mathbf{q}^n = \mathbf{q}^{n-1} + \mathbf{\Delta q}^n \ , \tag{14}$$

The iterations are terminated after reaching a predefined accuracy (ε_G), that is,

$$\left\|\mathbf{G}(\mathbf{q}^n, \lambda^n)\right\| \le \varepsilon_G \ , \tag{15}$$

On the basis of carried out numerical experiments, it was determined that the accuracy should satisfy $\varepsilon_G \in \left(0.1^{-3}, 0.1^{-6}\right)$.

This method allows us to analysis of the static behavior of a physically and geometrically nonlinear structural element, which was described by system of dynamic equilibrium equations after applying the critical damping.

Figures 2 and 3 schematically show load and displacement in pseudo time function for the dynamic relaxation method before and after applying critical damping. In the elastic range using the critical damping allows determining the displacement values that correspond to the equilibrium state of the undamped vibrations. Whereas in inelastic range, the displacements determined after introducing critical damping are usually lower than displacement values corresponding to the equilibrium state of the undamped vibration.

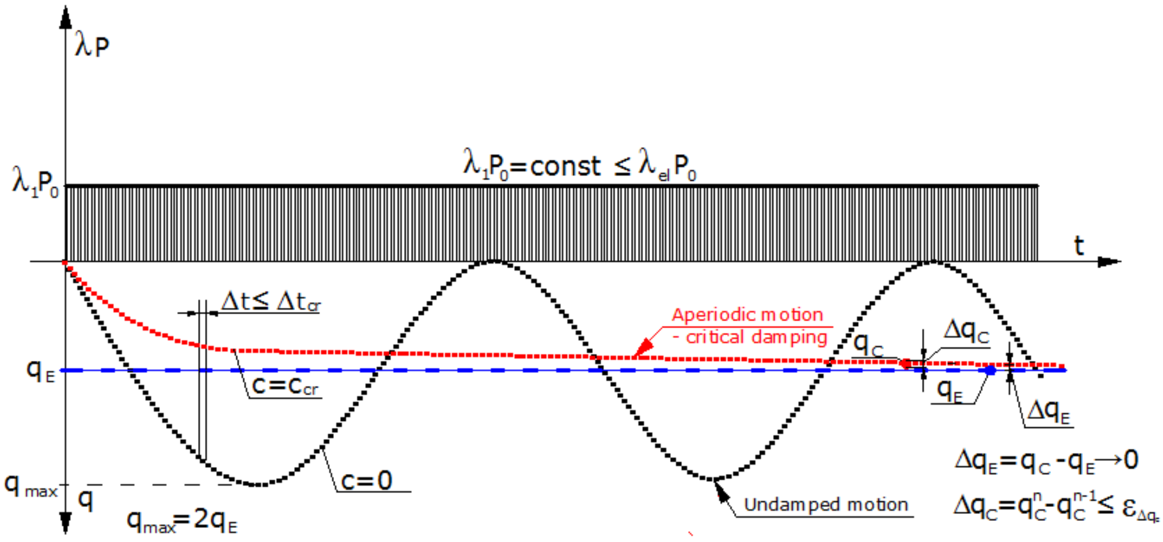

Fig. 2 Load and displacement vs. pseudo time for the dynamic relaxation method with critical damping in elastic range

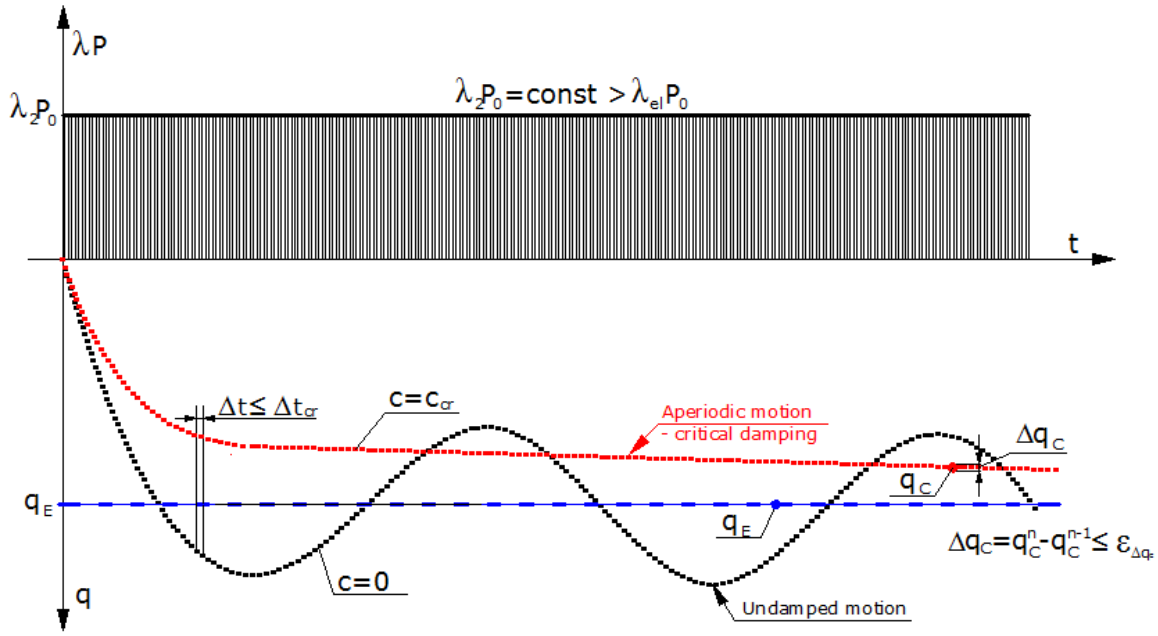

Fig. 3 Load and displacement vs. pseudo time for the dynamic relaxation method with critical damping in inelastic range

5. CONCLUSIONS

The paper presents a behavior analysis method for the bending and compression of reinforced concrete elements subjected to short-term static loads. The structural system analysis technique was developed using assumptions from the finite difference method. The central axis of the structural element was discretized, and the equilibrium equations and geometrical relationships in differential form were written. A joint-action rule was determined for the active concrete and steel layers of the discretized cross-sectional model.

A dynamic process that can describe the static problem by introducing critical damping, was analyzed. When this procedure is used to solve a system of nonlinear equilibrium equations it is referred to as the DRM. Using DRM, the equations of motion can be recursively solved in subsequent pseudo-time instants, for each node of the spatial division. There was no need to solve the system of algebraic equilibrium equations. The DRM was improved by introducing an additional constraint equation to the system of equilibrium equations, according to the assumptions of the arc length method. This method is called the DRM+AL, and can be used to solve the nonlinear equilibrium equations in the post-critical range.

When modeling the behavior of structural materials, the reduced plane compressive/tensile states with shear stress was considered. Equations derived from the theory of moderately large displacements in a bar system form the basis of structural behavior description, assuming infinitesimal deformations. This assumption is appropriate when describing strain and stress states in the concrete and steel layers of commonly used reinforced concrete elements.

The verification of developed method, numerical tests on a reinforced concrete beam and eccentrically loaded column**,** were carried out and indicate to high effectiveness of the method for a post-critical analysis.

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