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# On flow laws and constitutive relations in non-smooth elastoplasticity

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## ABSTRACT

In the present paper a formulation of flow laws and constitutive relations in non-smooth elastoplasticity is presented. The treatment refers to general non-smooth elastoplasticity problems and to problems characterized by non-smooth yield criteria and non-differentiable functions. The mathematical tools of subdifferential calculus are suitably adopted in order to supply proper mathematical instruments able to deal with non-smooth problems and non-differentiable functions. Generalized formulations of flow laws and loading/unloading conditions in non-smooth elastoplasticity are derived and presented within the mentioned theoretical framework. Connections between the proposed mathematical treatment and the classical relations in elastoplasticity are illustrated and discussed. The presented treatment is endowed with considerable advantages since it is well suited for the development of variational formulations and computational algorithms for structural problems in non-smooth elastoplasticity.

## 1. INTRODUCTION

In elastoplasticity the use of non-differentiable functions is required for the nature of the model problem which is characterized by loading/unloading conditions and singularities in the yield surfaces. The use of multivalued operators and non-differentiable functions leads to a non-standard treatment of typically non-smooth multisurface plasticity problems, for an analytical point of view see e.g. Koiter (1953), Koiter (1960) and Mandel (1965), and for a numerical point of view among others see e.g. Simo et al. (1988), Alfano et al (2001), De Angelis and Cancellara (2017) and De Angelis and Taylor (2015, 2016). In the present paper the evolutive elastoplasticity problem is formulated by adopting the tools and concepts of subdifferential calculus which is characterized by the necessary features for treating typically non-smooth problems. In generalized elastoplasticity the appropriate mathematical tools for

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describing constitutive relations among state and associated variables are those of subdifferential calculus, see e.g. Rockafellar (1970), Hiriart-Urruty and Lemarechal (1993) and Moreau (1973). Consequently in the present paper generalized elastoplastic constitutive relations and plasticity laws are illustrated and discussed in detail within the adopted mathematical framework. The presented generalized framework also shows to be appropriate from a variational point of view, since it can be illustrated that this approach is convenient and advantageous also for the development of variational formulations of structural problems in elastoplasticity.

## 2. THE CONTINUUM MODEL WITH INTERNAL VARIABLES

Let us consider a body $B$ in the reference configuration $\Omega\subset R^n$, with $1\leq n \leq 3$. Let $T \subset R_+$ be the time interval of interest, $\boldsymbol{V}$ the space of displacements, $\boldsymbol{D}$ the strain space and $\boldsymbol{S}$ the dual stress space. The displacement is denoted by $\mathbf{u}$: $\Omega \times T \rightarrow \boldsymbol{V}$ and the stress tensor by $\boldsymbol{\sigma} : \Omega\times T \rightarrow \boldsymbol{S}$. The strain tensor compatible with $\mathbf{u}$ is denoted by $\boldsymbol{\varepsilon} = \nabla^s\mathbf{u}$: $\Omega \times T \rightarrow \boldsymbol{D}$, where $\nabla^s$ is the symmetric part of the gradient. A small deformation theory is also assumed with quasi-static deformations, see e.g. Skrzypek and Hetnarski (1993), Duvaut and Lions (1992) and Lemaitre and Chaboche (1990). An elastic material behavior is considered with the convex conjugate potentials representing the elastic energy $W : \boldsymbol{D}\rightarrow R$ and the complementary elastic energy $W^* : \boldsymbol{S}\rightarrow R$. In linear elasticity they are expressed as

$$W(\boldsymbol{\varepsilon}^e)=\tfrac{1}{2}\langle\mathbf{C}\boldsymbol{\varepsilon}^e,\boldsymbol{\varepsilon}^e\rangle, \qquad W^*(\boldsymbol{\sigma})=\tfrac{1}{2}\langle\boldsymbol{\sigma},\mathbf{C}^{-1}\boldsymbol{\sigma}\rangle, \tag{1}$$

where $\boldsymbol{\varepsilon}^e$ is the elastic strain, $\langle\cdot,\cdot\rangle$ indicates a non-degenerate bilinear form acting on dual spaces and $\mathbf{C}$ denotes the elastic stiffness. The relations

$$\boldsymbol{\sigma}=\mathrm{d}W(\boldsymbol{\varepsilon}^e), \qquad \boldsymbol{\varepsilon}^e=\mathrm{d}W^*(\boldsymbol{\sigma}) \tag{2}$$

or equivalently

$$W(\boldsymbol{\varepsilon}^e)+W^*(\boldsymbol{\sigma})=\langle\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^e\rangle \tag{3}$$

apply to conjugate pairs $\{\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^e\}$ satisfying the elastic constitutive relation. We also express the plastic strain with $\boldsymbol{\varepsilon}^p = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^e$. In elastoplasticity with hardening behavior we introduce a dual pair of kinematic internal variables

$$\boldsymbol{\alpha}=(\boldsymbol{\alpha}_{kin}, \alpha_{iso}) \in \boldsymbol{X^*}=\boldsymbol{X}\times R \tag{4}$$

and static internal variables

$$\boldsymbol{\chi}=(\boldsymbol{\chi}_{kin}, \chi_{iso}) \in \boldsymbol{X'^*}=\boldsymbol{X'}\times R, \tag{5}$$

where $\alpha_{iso} \in R$ and $\chi_{iso} \in R$ represent isotropic hardening behavior, and $\boldsymbol{\alpha}_{kin} \in \boldsymbol{X}$ and $\boldsymbol{\chi}_{kin} \in \boldsymbol{X'}$ represent kinematic hardening behavior, $\boldsymbol{X}$ and $\boldsymbol{X'}$ being dual spaces.

We now introduce the hardening potential $H(\boldsymbol{\alpha})$ and its conjugate complementary hardening potential $H^*(\boldsymbol{\chi})$. The relations

$$\boldsymbol{\chi}=\mathrm{d}H(\boldsymbol{\alpha}), \qquad \boldsymbol{\alpha}=\mathrm{d}H^*(\boldsymbol{\chi}) \tag{6}$$

or equivalently

$$H(\boldsymbol{\alpha})+ H^*(\boldsymbol{\chi})=\langle\boldsymbol{\chi}, \boldsymbol{\alpha}\rangle \tag{7}$$

apply to conjugate pairs $\{\boldsymbol{\chi}, \boldsymbol{\alpha}\}$.
The hardening potentials are assumed to be expressed in decoupled form and for linear hardening they are represented by

$$H(\boldsymbol{\alpha})=½\mathbf{H}_{kin}\boldsymbol{\alpha}_{kin}\cdot\boldsymbol{\alpha}_{kin}+½\mathrm{H}_{iso}\alpha_{iso}^2 \tag{8}$$

and

$$H^*(\boldsymbol{\chi})=½\boldsymbol{\chi}_{kin}\cdot\mathbf{H}_{kin}^{-1}\boldsymbol{\chi}_{kin}+½\mathrm{H}_{iso}^{-1}\chi_{iso}^2, \tag{9}$$

resulting $\boldsymbol{\chi}_{kin}=\mathbf{H}_{kin}\boldsymbol{\alpha}_{kin}$ and $\chi_{iso}=\mathrm{H}_{iso}\alpha_{iso}$, and where $\mathbf{H}_{kin}$ and $\mathrm{H}_{iso}$ respectively represent the kinematic and isotropic hardening moduli. The static and kinematic internal variables are related by the equation $\boldsymbol{\chi}=\mathbf{H}\boldsymbol{\alpha}$, where $\mathbf{H}=\mathrm{diag}\,[\mathbf{H}_{kin}, \mathrm{H}_{iso}]$ is the hardening matrix. For a comparative analysis of linear and nonlinear kinematic hardening rules in elastoplasticity see e.g. De Angelis (2012a).
The elastic domain $\boldsymbol{\Sigma}$ is introduced by means of a convex yield function $f(\boldsymbol{\sigma}, \boldsymbol{\chi})$, such that

$$\boldsymbol{\Sigma} =\{(\boldsymbol{\sigma},\boldsymbol{\chi}) \in \boldsymbol{S}\times\boldsymbol{X'^*}: \quad f(\boldsymbol{\sigma},\boldsymbol{\chi}) \le 0\}. \tag{10}$$

Accordingly the convex sets

$$\boldsymbol{\Sigma}_{\sigma} = \{\boldsymbol{\sigma} \in \boldsymbol{S} : \quad (\boldsymbol{\sigma}, \boldsymbol{\chi}) \in \boldsymbol{\Sigma}\} \tag{11}$$

and

$$\boldsymbol{\Sigma}_{\chi} = \{\boldsymbol{\chi} \in \boldsymbol{X'^*}: \quad (\boldsymbol{\sigma}, \boldsymbol{\chi}) \in \boldsymbol{\Sigma}\} \tag{12}$$

represent the sections of the elastic domain at constant $\boldsymbol{\chi}$ and at constant $\boldsymbol{\sigma}$.
For the treatment of elastoplasticity with internal variables we consider the generalized standard material model introduced by Halphen and Nguyen (1975), where strains and kinematic internal variables, as well as stress and static internal variables, are gathered in suitably defined generalized variables. The generalized strain is represented by $\boldsymbol{\varepsilon}^*= (\boldsymbol{\varepsilon}, \mathbf{0})$, the generalized elastic and plastic strains are represented by $\boldsymbol{\varepsilon}^{e*}= (\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha})$ and $\boldsymbol{\varepsilon}^{p*}= (\boldsymbol{\varepsilon}^p, -\boldsymbol{\alpha})$ and the generalized stress is represented by $\boldsymbol{\sigma}^*=(\boldsymbol{\sigma}, \boldsymbol{\chi})$. The duality products between generalized variables are therefore expressed by

$$\langle \boldsymbol{\sigma}^*, \boldsymbol{\varepsilon}^* \rangle = \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \rangle, \quad \langle \boldsymbol{\sigma}^*, \boldsymbol{\varepsilon}^{e*} \rangle = \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}^{e} \rangle + \langle \boldsymbol{\chi}, \boldsymbol{\alpha} \rangle \tag{13}$$

and

$$\langle \boldsymbol{\sigma}^*, \boldsymbol{\varepsilon}^{p*} \rangle = \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}^{p} \rangle - \langle \boldsymbol{\chi}, \boldsymbol{\alpha} \rangle, \tag{14}$$

where the same symbols have been adopted for denoting duality products on different pairs of vector spaces.

## 3. FLOW LAWS AND LOADING/UNLOADING CONDITIONS IN NON-SMOOTH ELASTOPLASTICITY

The maximum plastic dissipation principle is formulated by, see e.g. Hill (1950),

$$D^p(\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}}) = \sup_{(\boldsymbol{\tau}, \mathbf{q}) \in \boldsymbol{\Sigma}} \{\langle \boldsymbol{\tau}, \dot{\boldsymbol{\varepsilon}}^p \rangle - \langle \mathbf{q}, \dot{\boldsymbol{\alpha}} \rangle\}, \tag{15}$$

where $(\boldsymbol{\tau}, \mathbf{q})$ denotes a generic generalized stress state, $(\boldsymbol{\sigma}, \boldsymbol{\chi})$ denotes the generalized stress state at the solution and $(\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}})$ is the given generalized plastic strain rate. We can usefully consider the plasticity problem as a convex optimization problem and the optimality conditions of the optimization problem are usefully adopted in the sequel for deriving a consistent formulation of the plasticity laws, see e.g. De Angelis (2000, 2007a, 2007b). For a comprehensive account see also De Angelis (2018) and De Angelis and Meola (2021).
In a model of non-smooth elastoplasticity with internal variables the maximum plastic dissipation principle is equivalently formulated by

$$D^p(\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}}) = \sup_{(\boldsymbol{\tau}, \mathbf{q}) \in \mathbf{S} \times \mathbf{X}'^*} \{\langle \boldsymbol{\tau}, \dot{\boldsymbol{\varepsilon}}^p \rangle - \langle \mathbf{q}, \dot{\boldsymbol{\alpha}} \rangle - U_{\boldsymbol{\Sigma}}(\boldsymbol{\tau}, \mathbf{q})\}, \tag{16}$$

where $U_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma}, \boldsymbol{\chi})$ represents the indicator function of the convex elastic domain defined by, see e.g. Rockafellar (1970) and Hiriart-Urruty and Lemarechal (1993),

$$U_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma}, \boldsymbol{\chi}) = \{0 \text{ if } (\boldsymbol{\sigma}, \boldsymbol{\chi}) \in \boldsymbol{\Sigma}; \quad +\infty \text{ if } (\boldsymbol{\sigma}, \boldsymbol{\chi}) \notin \boldsymbol{\Sigma}\}. \tag{17}$$

In non-smooth plasticity the evolutive laws are formulated by expressing that the generalized plastic flow belongs to the normal cone $N_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma}, \boldsymbol{\chi})$ to the elastic domain $\boldsymbol{\Sigma}$ at $(\boldsymbol{\sigma}, \boldsymbol{\chi})$ and accordingly it is

$$(\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}}) \in N_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma}, \boldsymbol{\chi}). \tag{18}$$

We now observe that $N_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma}, \boldsymbol{\chi})$ coincides with the subdifferential of the indicator function to the convex elastic domain $\boldsymbol{\Sigma}$ at $(\boldsymbol{\sigma}, \boldsymbol{\chi})$, and it results

$$N_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma}, \boldsymbol{\chi}) = \partial U_{\boldsymbol{\Sigma}}(\boldsymbol{\sigma}, \boldsymbol{\chi}). \tag{19}$$

Accordingly the evolutive law in elastoplasticity can be expressed equivalently in the subdifferential form

$$(\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}}) \in \partial U_\Sigma(\boldsymbol{\sigma}, \boldsymbol{\chi}), \tag{20}$$

which is formulated in components by

$$\dot{\varepsilon}^p \in \partial_\sigma\, U_{\Sigma\sigma}\, (\boldsymbol{\sigma}, \boldsymbol{\chi}) \tag{21}$$
$$-\,\dot{\alpha} \in \partial_\chi\, U_{\Sigma\chi}\, (\boldsymbol{\sigma}, \boldsymbol{\chi}) \tag{22}$$

and they express respectively the flow law of the plastic strain and the evolutive law of the kinematic internal variables.
We now introduce the conjugate $U_\Sigma^*(\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}})$ of the indicator function $U_\Sigma(\boldsymbol{\sigma}, \boldsymbol{\chi})$ as

$$U_\Sigma^*(\dot{\boldsymbol{\varepsilon}}^p,-\dot{\boldsymbol{\alpha}}) = \sup_{(\boldsymbol{\tau},\, \mathbf{q}) \in \mathbf{S} \times \mathbf{X'^*}}\{<\boldsymbol{\tau}, \dot{\boldsymbol{\varepsilon}}^p>-<\mathbf{q}, \dot{\boldsymbol{\alpha}}>-U_\Sigma(\boldsymbol{\tau}, \mathbf{q})\}, \tag{23}$$

which is equivalently expressed by

$$U_\Sigma^*(\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}}) = \sup_{(\boldsymbol{\tau},\, \mathbf{q}) \in \boldsymbol{\Sigma}} \{<\boldsymbol{\tau}, \dot{\boldsymbol{\varepsilon}}^p> - <\mathbf{q}, \dot{\boldsymbol{\alpha}}>\}, \tag{24}$$

and it coincides with the plastic dissipation, that is $U_\Sigma^*(\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}}) = D^p\,(\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}})$. The flow rule (20) is therefore expressed in the equivalent inverse subdifferential form

$$(\boldsymbol{\sigma}, \boldsymbol{\chi}) \in \partial U_\Sigma^*(\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}}), \tag{25}$$

and relations (20) and (25) can be formulated equivalently in the Fenchel's form

$$U_\Sigma(\boldsymbol{\sigma}, \boldsymbol{\chi}) + U_\Sigma^*(\dot{\boldsymbol{\varepsilon}}^p, -\dot{\boldsymbol{\alpha}}) = <\boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}}^p> - <\boldsymbol{\chi}, \dot{\boldsymbol{\alpha}}>. \tag{26}$$

The illustrated formulation for the evolutive laws in non-smooth elastoplasticity is useful since it shows to be capable to provide a complete variational formulation in plasticity and rate plasticity problems, see e.g. De Angelis (2000) and De Angelis and Cancellara (2017). A description of the consequences of different loading programs on the inelastic behavior of solid materials has been reported e.g. by De Angelis (2012b, 2013, 2015) and De Angelis et al. (2018).
For a treatment of generalized evolutive laws in plasticity and viscoplasticity see e.g. De Angelis and Cancellara (2017), De Angelis (2018), De Angelis and Meola (2021). Such approach is also advantageous for the derivation of variational formulations which form the basis for the development of numerical algorithms in finite element applications, see e.g. Alfano et al. (2001) and De Angelis and Taylor (2015, 2016). Extension of the treatment to cases with different material models can be studied, see e.g. De Angelis and De Angelis (2021). Further extensions of this treatment can also be

investigated in order to include the behaviour of materials with voids see e.g. De Cicco and De Angelis (2020).

## 4. CONCLUSIONS

A formulation has been presented for a consistent derivation of generalized flow laws and constitutive relations in non-smooth elastoplasticity with internal variables. The formulation holds for non-smooth elastoplasticity and it is capable to treat problems characterized by non-smooth yield criteria and problems characterized by non-differentiable functions such as the complementarity loading/unloading conditions. The proper mathematical tools of subdifferential calculus have been used since they supply the suitable mathematical instruments for dealing with non-smooth problems and non-differentiable functions. Generalized formulations of flow laws and loading/unloading conditions have been illustrated within the mentioned mathematical treatment. Equivalent formulations of the evolutive equations and of the loading/unloading conditions have been described. Moreover, the presented generalized treatment shows to be advantageous also from the point of view of a variational formulation of the model problem, since it is capable to supply a consistent variational treatment for structural problems in elastoplasticity.

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