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Stress concentration effects in chiral Cosserat elastic plates

*Simona De Cicco[1)] and Fabio De Angelis[2)]

[1), 2)] *Department of Structures for Engineering and Architecture,*
University of Naples Federico II,
via Forno Vecchio 36, 80134 Naples, Italy
[1)] *simona.decicco@unina.it*

ABSTRACT

This paper is concerned with the theory of chiral Cosserat elastic plates. In 1967 Eringen established a theory of isotropic achiral plates in the framework of Cosserat elasticity. This theory is based on the assumption that the microrotation vector does not vary across the thickness of the plate. De Cicco and Iesan (2013) extended to chiral Cosserat elastic plates without using this assumption. In contrast with the theory of achiral plates, the stretching and flexure in the chiral plates, cannot be treated independently of each other. In this paper we consider the problem of an infinite plate with a circular hole. The plate is subject to uniform pressure at infinity. We solve the problem in closed form.

1. INTRODUCTION

In recent years many researches have been devoted to the study of the mechanical behaviour of chiral materials. These investigations are motivated by the recent interest in the using the chiral elastic materials as a model for carbon nanotubes (Chandraseker, K., Mukherjee, 2006), bones (Park, H.C., Lakes, R.S., 1986), honeycombs structures (Prall, D., Lakes, R. S., 1997) as well as auxetic materials (Donescu, S., Chiroiu, V., Munteanu, L., 2009). The behaviour of these materials is strongly dependent by chirality and cannot be described by means of the classical theory of continua. Lakes (2001) presented some simple examples illustrating chirality in deformation of slabs and plates. In this paper we investigate a two-dimensional problem in the theory of chiral Cosserat solids. We study the deformation of a thin plate with a circular hole under constant pressure at infinity. In contrast with the case of achiral plates the stretching and flexure cannot be treated independently of each other (Iesan D. 2010). The problem under consideration is a typical application of the classical two-dimensional elasticity. Subsequently, the problem has been extended to non classical solids (De Cicco S. 2003, Iesan D. 2009, De Cicco S. 2014, De Cicco S., De Angelis F. 2019). The solution is significant in the analysis of structural fatique and is of crucial importance in the study of

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[1)] Professor
[2)] Professor

the behaviour of structural members with irregularities.
A theory of achiral Cosserat elastic plates was established by Eringen (1967). This theory is based on the assumption that the microrotation vector does not vary across the thickness of the plate. De Cicco and Iesan (2013) extended the theory to the case of chiral Cosserat elastic plates without using this assumption.
In Section 2 we present the basic equations of chiral Cosserat elastic solids. In Section 3 we summarize the field equations of the theory of thin plate. In Section 4 we study the deformation of a circular plate with a circular hole under uniform pressure at infinity. The main feature is that in the case of chiral plates, a uniform pressure acting on the boundary of the plate at infinity produces a microrotation of the material particles.

2. PRELIMINARIES

Let $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ be the displacement vector field and the microrotation vector field on a body *B*, respectively, we summarize the basic equations of the equilibrium theory for chiral Cosserat elastic solids

geometrical equations

$$e_{ij} = u_{j,i} + \varepsilon_{ijk}\varphi_k, \qquad \chi_{ij} = \varphi_{j,i} \tag{1}$$

equilibrium equations

$$t_{ji,j} + f_i = 0, \qquad m_{ji,j} + \varepsilon_{ijk}t_{jk} + g_i = 0 \tag{2}$$

constitutive equations

$$\begin{aligned} t_{ij} &= \lambda e_{rr}\delta_{ij} + (\mu+\kappa)e_{ij} + C_1\chi_{ss}\delta_{ij} + C_2\chi_{ji} + C_3\chi_{ij} \\ m_{ij} &= \alpha\chi_{rr}\delta_{ij} + \beta\chi_{ji} + \gamma\chi_{ij} + C_1e_{rr}\delta_{ij} + C_2e_{ji} + C_3e_{ij}. \end{aligned} \tag{3}$$

We have used the following notations: e_{ij} and χ_{rs} are strain tensors, ε_{ijk} is the alternating symbol, t_{ij} is the stress tensor, m_{ij} is the couple stress tensor, f_i is the body force, g_i is the body couple, $\lambda,\mu,\kappa,\alpha,\beta,\gamma,C_1,C_2$ and C_3 are constitutive constants and δ_{ij} is the Kronecker's delta.
As consequence of the positive definiteness of the elastic potential we deduce the following inequalities (see [6])

$$\begin{gathered} \lambda + 2\mu + \kappa > 0, \quad 2\mu + \kappa > 0 \quad \kappa > 0 \\ \gamma + \beta > 0 \quad \gamma - \beta > 0, \\ (\lambda + 2\mu + \kappa)(\alpha + \beta + \gamma) - (C_1 + C_2 + C_3)^2 > 0 \end{gathered} \tag{4}$$

We denote by t_i the surface force and by m_i the surface moment acting at a regular point of ∂B. The boundary conditions are given by

$$t_i = t_{ji}n_j, \quad m_i = m_{ji}n_j. \tag{5}$$

The equilibrium problem of an elastic chiral Cosserat body consists in the solving of the system of equations (1)-(3) with the boundary conditions (5).

3. CHIRAL COSSERAT PLATES

In the following we consider a right cylinder of isotropic material which occupies the region B. We denote by Σ the open cross-section, Γ the boundary of Σ and Π the

lateral boundary. The axis Ox_3 of our coordinate system is chosen in such a way that the plane x_1Ox_2 is the middle plane. The cylinder is assumed to be of length $2h$.
In the theory of chiral Cosserat elastic plates formulated by De Cicco-Iesan in [] the stretching and the flexure of plates cannot be treated independently of each other. The theory is based on the assumption that the displacement $\boldsymbol{u}$ and the microrotation $\boldsymbol{\varphi}$ assume the form

$$u_\alpha = w_\alpha(x_1,x_2) + x_3 v_\alpha(x_1,x_2), \quad u_3 = w_3(x_1,x_2)$$
$$\varphi_\alpha = \psi_\alpha(x_1,x_2) + x_3 \kappa_\alpha(x_1,x_2), \quad \varphi_3 = \psi_3(x_1,x_2) \tag{6}$$

In (6) w_α and ψ_3 characterize the extension and v_α, w_3 and ψ_α the flexure of the cylinder.
The equilibrium equations of chiral Cosserat elastic plates are given by

$$\begin{aligned} &\tau_{\alpha s,\alpha} + F_s = 0, \\ &\mu_{\beta\alpha,\beta} + \varepsilon_{3\rho\alpha}(\tau_{3\rho} + \tau_{\rho 3}) + G_\alpha = 0, \\ &\mu_{\alpha 3,\alpha} + \varepsilon_{3\rho\beta}\tau_{\rho\beta} + G_3 = 0, \end{aligned} \tag{7}$$

where we have used the notations

$$\begin{aligned} &\tau_{ij} = \frac{1}{2h}\int_{-h}^{h} t_{ij}dx_3, \quad \mu_{ij} = \frac{1}{2h}\int_{-h}^{h} m_{ij}dx_3, \\ &F_i = \frac{1}{2h}\left(\int_{-h}^{h} f_i dx_3 + [t_{3i}]_{-h}^{h}\right) \\ &G_i = \frac{1}{2h}\left(\int_{-h}^{h} g_i dx_3 + [m_{3i}]_{-h}^{h}\right) \end{aligned} \tag{8}$$

The equations (7) are obtained by integrating the equilibrium equations (2) with respect to x_3 over the thickness of the plate.
To the equilibrium equations (7) we must adjoin the following equations

$$\begin{aligned} &H\sigma_{\beta\alpha,\beta} - 2h\tau_{3\alpha} + H_\alpha = 0, \\ &H\pi_{\beta\alpha,\beta} + \varepsilon_{3\rho\alpha}H(\sigma_{3\rho} - \sigma_{\rho 3}) - 2h\mu_{3\alpha} + P_\alpha = 0 \end{aligned} \tag{9}$$

where

$$\begin{aligned} &\int_{-h}^{h} x_3 t_{ij} dx_3 = \frac{2}{3}h^3\,\sigma_{ij}\,, \\ &\int_{-h}^{h} x_3 m_{ij} dx_3 = \frac{2}{3}h^3\,\pi_{ij}\,, \\ &\int_{-h}^{h} x_3 f_\alpha dx_3 + [x_3 t_{3\alpha}] = H_\alpha, \\ &\int_{-h}^{h} x_3 g_\alpha dx_3 + [x_3 m_{3\alpha}] = P_\alpha, \quad H = \frac{2}{3}h^3 \end{aligned} \tag{10}$$

The equations (9) are obtained taking the cross product of the equations (2) with the vector $x_3\boldsymbol{e_3}$ where $\boldsymbol{e_3}$ is the unit outward normal to the plane x_1Ox_2, and integrating over the thickness of the plate.
The boundary conditions (5) are rewritten in the form

$$\tau_{\beta\kappa} n_\beta = \tau_\kappa^*, \quad \mu_{\beta\kappa} n_\beta = \mu_\kappa^*,$$
$$\sigma_{\beta\alpha} n_\beta = \sigma_\alpha^*, \quad \pi_{\beta\alpha} n_\beta = \pi_\alpha^* \quad on\ \Gamma \tag{11}$$

where τ_κ^*, μ_κ^*, σ_α^*, and π_α^* are prescribed functions.
In the case of two dimensional problem, we suppose that

$$w_3 = 0, \quad \psi_3 = 0. \tag{12}$$

Moreover, we consider null body loads

$$F_s = 0, \ \ G_s = 0, \ \ H_\alpha = 0, \ \ P_\alpha = 0. \tag{13}$$

The equations (7) and (9) can be written in terms of the functions w_α, ψ_α, ν_α and χ_α. We have

$$\begin{aligned}
&(\mu+\kappa)\Delta w_\alpha + (\mu+\lambda) w_{\beta,\beta\alpha} + C_3 \Delta\psi_\alpha + (C_1+C_2)\psi_{\beta,\beta\alpha} = 0\\
&\kappa\varepsilon_{3\alpha\beta}\psi_{\beta,\alpha} + \mu\nu_{\rho,\rho} + C_2\chi_{\rho,\rho} = 0\\
&C_3\Delta w_\alpha + (C_1+C_2) w_{\beta,\beta\alpha} + \gamma\Delta\psi_\alpha + (\alpha+\beta)\psi_{\beta,\beta\alpha} + \kappa\varepsilon_{3\beta\alpha}\nu_\beta -\\
&-2\kappa\psi_\alpha + (C_3-C_2)\varepsilon_{3\beta\alpha}\chi_\beta = 0\\
&\kappa\varepsilon_{3\beta\alpha} w_{\alpha,\beta} + 2\varepsilon_{3\alpha\beta}(C_3-C_2)\psi_{\beta,\alpha} + C_2\nu_{\alpha,\alpha} + \beta\chi_{\rho,\rho} = 0,\\
&H\left[(\mu+\kappa)\Delta\nu_\alpha + (\lambda+\mu)\nu_{\beta,\beta\alpha} + C_3\Delta\chi_\alpha + (C_1+C_2)\chi_{\rho,\rho\alpha}\right] -\\
&-2h\left[\kappa\varepsilon_{3\beta\alpha}\psi_\beta + (\mu+\kappa)\nu_\alpha + C_3\chi_\alpha\right] = 0\\
&H\left[\gamma\Delta\chi_\alpha + (\alpha+\beta)\chi_{\beta,\beta\alpha} + C_3\Delta\nu_\beta + (C_1+C_2)\nu_{\beta,\beta\alpha}\right] -\\
&-2H\,\kappa\chi_\beta - 2h\left[(C_2-C_3)\varepsilon_{3\beta\alpha}\psi_\beta + \gamma\chi_\beta + C_3\nu_\beta\right] = 0,
\end{aligned} \tag{14}$$

where Δ is the two-dimensional Laplacian.
Existence and uniqueness theorems of the solution of the system (14) has been presented by De Cicco and Iesan ().

A PLATE WITH A CIRCULAR HOLE

In this section we consider a circular infinite plate with a circular hole under uniform radial pressure at infinity. The boundary of the hole is supposed to be a stress free.
Let $r = (x_1^2 + x_2^2)^{1/2}$ and a be a positive constant. We assume that a is the radius of the hole and the region Σ is defined by $\Sigma = \{(x_1, x_2, x_3): x_1^2 + x_2^2 > a, \ x_3 = 0\}$. Let p be a given constant the boundary conditions are given by

$$\tau_k^* = 0, \quad \mu_k^* = 0, \quad \sigma_\alpha^* = 0, \ \ \pi_\alpha^* = 0 \quad \text{on}\ r = a$$
$$\tau_\beta = p n_\beta, \quad \mu_k^* = 0, \quad \sigma_\alpha^* = 0, \ \ \pi_\alpha^* = 0 \quad for\ r \to \infty \tag{15}$$

We introduce the unknown functions U, Ψ, S and Q satisfying the following equality

$$w_\alpha = U_{,\alpha}, \quad \psi_\alpha = \Psi_{,\alpha}, \quad \nu_\alpha = \varepsilon_{3\alpha\beta} S_{,\beta}, \quad \chi_\alpha = \varepsilon_{3\alpha\beta} Q_{,\beta}. \tag{16}$$

Using the relations (16) the system (14) reduces to

$$\Delta(\zeta_1 U + c\Psi) = 0$$
$$c\Delta u + \zeta_2(\Delta - s^2)\Psi + \kappa S + (C_3 - C_2) Q = 0$$

$$(\Delta - d^2)[(\mu+\kappa)S + C_3 Q] + \kappa d^2 \Psi = 0$$
$$C_3(\Delta - d^2)S + \gamma(\Delta - p^2)Q + d^2(C_3 - C_2)\Psi = 0 \tag{17}$$

where

$$s = [2\kappa/(\alpha+\beta+\gamma)]^{1/2}, \quad d = (2h/H)^{1/2}, \quad p = [(d^2\gamma + 2\kappa)/\gamma]^{1/2}$$
$$y_1 = \lambda + 2\mu + \kappa, \quad \zeta_2 = \alpha+\beta+\gamma, \quad c = C_1 + C_2 + C_3 \tag{18}$$

Let $V = V(r)$ be a function of class C^6, satisfying the equations

$$\Delta V = 0 \tag{19}$$

where D is the operator

$$D = \gamma_1(\Delta - K_1^2)(\Delta - K_2^2)(\Delta - K_3^2) \tag{20}$$

and

$$\gamma_1 = \beta_1 e_1 - 1, \quad \beta_1 = \frac{C_3}{\gamma}, \quad e_1 = \frac{C_3}{\mu+\kappa}. \tag{21}$$

In (20) the constants K_s are the roots of the equation

$$a_1 x^3 - a_2 x^2 + a_3 x + a_4 = 0 \tag{22}$$

where the coefficients a_s are given by

$$a_1 = \gamma_1, \quad a_2 = \gamma_1(q^2 + d^2) + \gamma_2,$$
$$a_3 = \gamma_1 q^2 d^2 + \gamma_2(q^2 + d^2) + \beta_3\gamma_3 + \frac{1}{2}q^2\gamma_3,$$
$$a_4 = \left[q^2\left(\frac{1}{2}\beta_2 e_1 - \gamma_2\right) - \beta_3(\beta_2 - \beta_1 e_2)\right]d^2 - \frac{1}{2}p^2 q^2 e_2.$$
$$\gamma_2 = \beta_1 e_1 d^2 - p^2, \quad \gamma_3 = e_2 - \beta_2 e_1,$$
$$\beta_2 = \frac{d^2(C_3 - C_2)}{\gamma}, \quad \beta_3 = \frac{\zeta_1(C_3 - C_2)}{d_1}, \quad e_2 = \frac{\kappa d^2}{\mu+\kappa}$$
$$q^2 = \frac{2\kappa\zeta_1}{d_1}, \quad d_1 = \zeta_1\zeta_2 - c^2\,. \tag{23}$$

The function V can be expressed as

$$V = B_i V_i \tag{24}$$

where B_i are arbitrary constants and the functions V_i satisfy the equation

$$(\Delta - K_i^2)V_i = 0 \tag{25}$$

[no sum; *i*=1,2,3].
The displacement and the stress must be finite at infinity. Under this condition the solution of equation (25) is given by

$$V_i = K_0(K_{ir}), \tag{26}$$

where K_n is the modified Bessel function of order n of the third kind. From (24) we have

$$V = \sum_{i=1}^{3} K_0(K_{ir}), \tag{27}$$

From (17) and (27) the functions Ψ, S and Q are explicitly determined

$$\Psi = \sum_{i=1}^{3} a_{1i} B_i K_0(K_{ir}), \quad S = \sum_{i=1}^{3} a_{2i} B_i K_0(K_{ir}),$$
$$Q = \sum_{i=1}^{3} a_{3i} B_i K_0(K_{ir}), \tag{28}$$

where

$$a_{1i} = \gamma_1 K_i^2(K_i^2 - d^2), \quad a_{2i} = (e_2 - \beta_2 e_1)K_i^2 - (e_2 p^2 - \beta_2 e_1 d^2),$$
$$a_{3i} = (\beta_2 - \beta_1 e_2)(K_i^2 - d^2) \tag{29}$$

The solution of $(17)_1$ gives

$$U = -\zeta\Psi + B_0 + B_4 ln\, r \quad (30)$$

where $\zeta = c/\zeta_1$, and B_0 and B_4 are arbitrary constants.
By imposing the boundary conditions (15) we determine the constants B_s and the problem is solved.

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