The 2021 World Congress on
Advances in Structural Engineering and Mechanics (ASEM21)
GECE, Seoul, Korea, August 23-26, 2021

Analytical technique of moment-curvature response of steel fibre-reinforced concrete beams

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ABSTRACT

This paper is focused on the development of an analytical technique for determining the moment-curvature response of steel fibre-reinforced concrete (SFRC) beams with conventional longitudinal reinforcement. Prevailing formula of residual tensile stress of SFRC in the post-crack regime is adapted and modified to form the tensile stress block of SFRC section under bending, with the introduction of the effective residual strain concept. This circumvents the need of conducting dedicated mechanical test for individual SFRC members. The tension stiffening phenomenon is accounted for by the reinforcement-related approach, and the tension stiffening force is assessed by considering the equivalent reinforced concrete beam having the same geometrical characteristics. To validate the proposed analytical technique, the moment-curvature response of SFRC beam specimens with different steel fibre volumes and reinforcement ratios are analysed. The computed results show very good agreement with the experimentally obtained results. The authors advocate the use of the proposed technique in deformational analysis of SFRC flexural members.

1. INTRODUCTION

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The combination of steel fibre with concrete to form steel fibre-reinforced concrete (SFRC) is a proven solution to improve post-crack resistance, ductility and deformability of concrete elements. The steel fibres can prevent crack proliferation by transferring tensile stresses across the cracks to tension along the fibres and to the bond between the fibres and concrete (Kovács and Balázs 2004; Singh 2017). Such crack bridging mechanism increases the energy absorption in the post-crack regime of SFRC, and this can particularly enhance the flexural ductility of SFRC beams. The structural behaviour of SFRC beams has been vastly researched. For example, Adebar et al. (1997) tested SFRC beams with difference fibre volume and geometry, and reported that the shear strength of beams increased with fibre volume, and the ductility increased with fibre length. Barros and Figueiras (1999) conducted experimental testing of SFRC beams and slabs, and proposed a layered model for prediction of the flexural behaviour from the cross-section characteristics. Cuchiara et al. (2004) tested SFRC beams containing hooked fibres and found that the use of appropriate fibre volume could lead to ductile failure, instead of relying on the provision of shear reinforcement to avoid brittle failure. De Montaignac et al. (2012) performed load tests of rectangular and T-shaped SFRC beams and suggested a design model of crack width based on the test results.

Design approaches and test methodologies of engineering properties of SFRC elements have been promulgated in a number of prevailing design codes and standards (ACI Committee 544 1988; RILEM TC 162-TDF 2000, 2001, 2002, 2003; CEN 2005; DAfStb 2012). RILEM TC 162-TDF (2000, 2001, 2002, 2003) recommended the residual stress-strain design method to account for the post-crack resistance of SFRC. Uniaxial tension test of SFRC element (RILEM TC 162-TDF 2001) and three-point bending test of SFRC notched beam specimen (RILEM TC 162-TDF 2000; CEN 2005) have been specified in the relevant RILEM standards and by the European Committee for Standardization. According to European Standard EN 14651 (CEN 2005), the limit of proportionality and residual flexural tensile stress at different crack mouth opening displacement (CMOD) can be obtained from the three-point bending test. The German DAfStb Guideline (DAfStb 2012) recommended another version of bending test: four-point bending test of SFRC un-notched beam specimen. The loading is applied at third points of the specimen span and the residual tensile stress-strain relationship is obtained from the load-deflection diagram of specimen.

A variety of analytical models were proposed to facilitate structural analysis and design of SFRC flexural members. Lok and Pei (1998) and Lok and Xiao (1999) derived nonlinear tension softening models of SFRC under uniaxial tension and flexure, taking into account the mechanical properties of concrete, fibre volume and aspect ratio, and the bond stress between fibre and concrete. Naaman (2003) devised a widely-used semi-empirical formula for determination of the residual tensile stress. The formula is able to reflect, inter alia, the effects of fibre volume and geometry. Campione and his co-workers (Campione et al. 2006; Campione 2008) developed semi-empirical models for evaluating the flexural and shear behaviour of SFRC beams with shear reinforcement. In considering the flexural behaviour, perfect bond between reinforcement and concrete was assumed and the tension stiffening effect of cracked concrete was excluded. However, there have been concerns from engineers and researchers regarding the ease of application and accuracy of the existing models.

In this paper, the authors aim to develop a new analytical technique for SFRC beams with conventional longitudinal reinforcement, with a view to improve calculation accuracy and facilitate practical applications. Details of the proposed technique are described hereunder.

2. METHODOLOGY OF DEFORMATIONAL ANALYSIS

2.1 Assumptions and Governing Equations

In the analytical derivation, the following general assumptions have been made:

- For both uncracked and cracked sections of SFRC beam, plain sections remain plane upon bending and the strain distribution along the depth of section is linear
- Both concrete under compression and reinforcement behave as linearly elastic materials (this assumption is close to reality in the service regime)
- The tension stiffening stress is attributed to the reinforcement (this is also referred to as reinforcement-related approach of tension stiffening)
- At given level of strain in the tensile reinforcement at the cracked section, the tension stiffening force of SFRC element and that of conventional reinforced concrete (RC) element are equal (during the analysis process, the equivalent RC beam is defined as a geometrically identical beam with the steel fibres omitted)
- Within the tension zone of a SFRC beam, the stress block is rectangular with constant residual stress up to the neutral axis
- At the crack formation stage, the intensity of stress block increases linearly with the external load from zero to the residual stress value, thereafter the intensity is constant at the stage of stable cracks

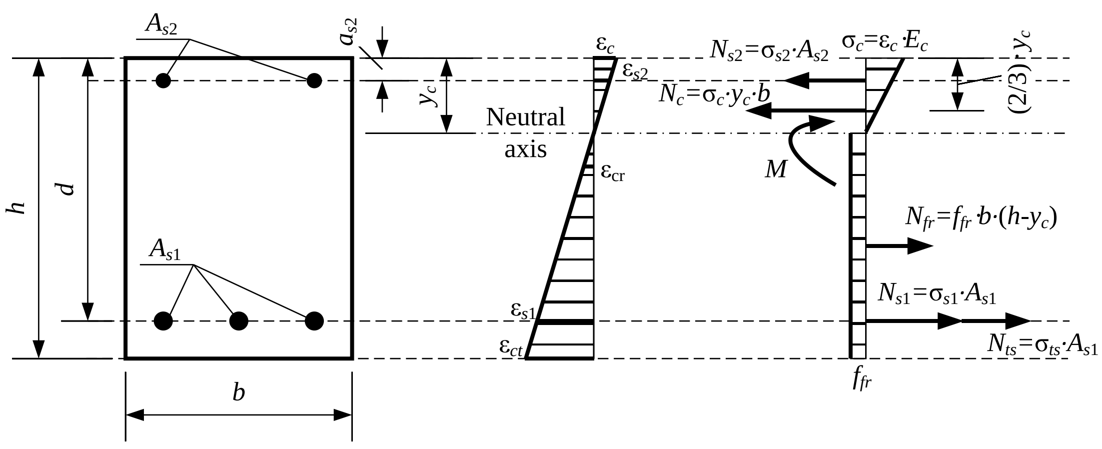

Fig. 1 Doubly reinforced section subjected to external bending

A doubly reinforced section subjected to external bending as shown in Fig. 1 is considered. With reference to Fig. 1 for the meaning of mathematical symbols, the equilibrium equations of axial forces and bending moments are:

$$N_c + N_{s2} - N_{fr} - N_{s1} - N_{ts} = 0 \quad (1)$$

$$M + N_{s2}\left(\frac{y_c}{3} - a_{s2}\right) - N_{fr}\left(\frac{h-y_c}{2} + \frac{2}{3}y_c\right) - (N_{s1} + N_{ts})\left(d - \frac{y_c}{3}\right) = 0 \quad (2)$$

In the above, N_{ts} denotes the tension stiffening force, and N_{fr} denotes the residual tensile force, which is the product of residual stress of SFRC f_{fr} and area of tension zone in the cross-section. The value of f_{fr} is dependent on a number of parameters, and will be discussed in the subsequent sub-section. The equilibrium equations can be expressed in terms of stresses and re-arranged to become:

$$0.5by_c\sigma_c + A_{s2}\sigma_{s2} - f_{fr}b(h - y_c) - A_{s1}\sigma_{s1} - N_{ts} = 0 \quad (3)$$

$$M = (N_{ts} + A_s\sigma_s)\left(d - \frac{y_c}{3}\right) - A_{s2}\sigma_{s2}\left(\frac{y_c}{3} - a_{s2}\right) + f_{fr}b(h - y_c)\left(\frac{h-y_c}{2} + \frac{2}{3}y_c\right) \quad (4)$$

By the plain section hypothesis, the following equations of strains can be established, in which ε_{sm} is the mean strain of tension reinforcement:

$$\varepsilon_c = \frac{y_c}{(d-y_c)}\varepsilon_{sm} \quad (5)$$

$$\varepsilon_{s2} = \frac{(y_c-a_{s2})}{(d-y_c)}\varepsilon_{sm} \quad (6)$$

Treating Eq. (3) to Eq. (6) as simultaneous equations and making the neutral axis depth y_c as the subject of equation, a quartic equation with y_c as the unknown can be set up. The quartic equation can be solved by algebraic means (Neumark 1965). For brevity, the mathematical expression of the solution is omitted here. Readers may refer to the literature for details (Neumark 1965; Kaklauskas et al. 2021). Among the four roots, the two imaginary roots and one negative real root should be rejected, and the remaining positive real root is accepted as the solution of y_c.

2.2 Residual Stress of SFRC

The residual stress of SFRC is dependent on the concrete and the steel fibre properties which can vary from case to case. In theory, dedicated mechanical testing needs to be conducted to obtain the value of f_{fr} on a case by case basis. To simplify the process, the design formula proposed by Naaman (2003) may be used:

$$f_{fr} = \lambda\tau\, V_f\, \frac{l}{d} \quad (7)$$

In Eq. (7), λ is a product of several coefficients to account for the fibres characters. Naaman (2003) suggested that $\lambda = \lambda_1\lambda_2\lambda_3$, where λ_1, λ_2, and λ_3 are respectively coefficients to reflect the influence of fibre pull-out length, orientation of fibre, and number of fibres per unit area, and they have the values of 0.25, 1.2, and 1.0 in normal circumstances; while Campione et al. (2006) suggested that $\lambda = \lambda_1\lambda_2\lambda_3\lambda_4$, where λ_1, λ_2, λ_3 are the same as aforementioned and λ_4 is a coefficient to reflect the bond characteristics of different fibre types, and it has the value of 0.5, 0.75 or 1.0 for rounded fibres, crimped fibres or hooked fibres, respectively. For the remaining symbols, τ is the average bond stress developed in a single fibre embedded in concrete, and it is taken as two times the tensile strength of concrete in this study; l, d and V_f are the length, diameter and volume fraction of fibres, respectively. In this study, the prevailing formula by Naaman (2003) and the modification of λ by Campione et al. (2006) are adopted.

From the last assumption in the preceding sub-section, during the stage of stable cracks, the intensity of tensile stress block is taken as the residual stress f_{fr}, whereas during the crack formation stage, the intensity should be linearly interpolated with respect to the bending moment. Mathematically, the stress block intensity is given by $f_{fr,ef} = [(M - M_{cr})/(\zeta - 1)M_{cr}]f_{fr}$, where M is the external bending moment, M_{cr} is the cracking moment, ζ is the ratio of stabilised cracking moment to initial cracking moment and is taken as 2.0 in this study. The notation $f_{fr,ef}$ can be termed the effective residual stress, which is a novel concept in SFRC design accompanying the proposed analytical technique.

2.3 Analysis Procedures

The moment-curvature response of SFRC beams is obtained by means of deformational analysis. The external bending moment is divided into finite incremental steps in order to capture the changes in flexural stiffness due to cracking. At every load step, the curvature is deterministic. The solution process can be readily automated in excel spreadsheet and is simple to implement by engineers. The analysis procedures consist of the following steps:

Firstly, for a given external bending moment M, equilibrium and compatibility equations are set up (c.f. Eq. (3) to Eq. (6)) with the assumption of $N_{ts} = 0$. The neutral axis depth is evaluated as the positive real root of the quartic equation of y_c. At the cracked section, the tensile strain of steel ε_{s1} is denoted by ε_{si}. By back-substituting y_c into Eq. (5) or (6), the value of ε_{si} can be obtained as ε_{sm} (tension stiffening is assumed to be non-existent).

Secondly, the equivalent RC beam is considered (by way of the fourth assumption above), as illustrated in Fig. 2. The bending moment M_{RC} for the equivalent RC beam is determined based on Eq. (8), where I_{cr} is the moment of inertia of the cracked equivalent RC beam section evaluated from Eq. (9).

$$M_{RC} = \varepsilon_{si} \frac{E_c I_{cr}}{(d - y_c)} \tag{8}$$

$$I_{cr} = \frac{E_c {y_c}^3 b}{3} + (E_{s2} - E_c)(y_c - a_{s2})^2 A_{s2} + E_{s1}(d - y_c)^2 A_{s1} \tag{9}$$

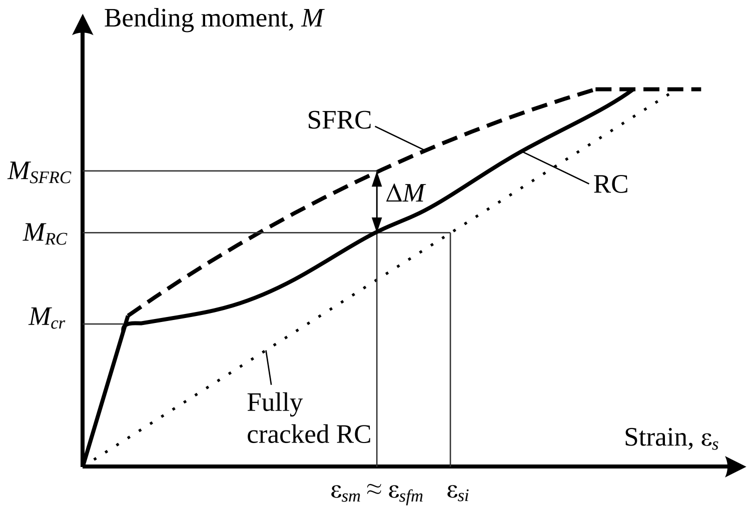

Fig. 2 Relationship between moment and steel strain for SFRC and equivalent RC beams

Thirdly, the tension stiffening force N_{ts} of the equivalent RC beam is computed according to Eq. (10). Without the residual stress of SFRC comes into play (the equivalent RC beam is considered) and with the adoption of reinforcement-related approach of tension stiffening, the value of y_c can be obtained as the positive real root of the relevant cubic equation (Kaklauskas et al. 2016, 2021; Kaklauskas and Gribniak 2016) (the full mathematical derivations were contained in these publications and are omitted here).

$$N_{ts} = \kappa_m \left[\frac{E_c {y_c}^2 b}{2A_{s1}} + (E_{s2} - E_c)(y_c - a_{s2}) \frac{A_{s2}}{A_{s1}} - E_{s1}(d - y_c) \right] A_{s1} \tag{10}$$

where κ_m is the mean curvature corresponding to M_{RC} and is given by:

$$\kappa_m = (1 - \xi)\kappa_{el} + \xi \kappa_{cr} = (1 - \xi) \frac{M}{E_c I_{el}} + \xi \frac{M}{E_c I_{cr}} \tag{11}$$

Eq. (11) is actually in accordance with the provision in fib Model Code 2010 (fib 2013) that describes the interpolation technique of the deformation parameter, with ξ being the distribution coefficient as determined from Eq. (12). I_{el} denotes the moment of inertia of the uncracked equivalent RC beam section ($I_{el} = bh^2/6$).

$$\xi = 1 - \left(\frac{M_{cr}}{M} \right)^2 \tag{12}$$

Fourthly, with respect to the calculated value of N_{ts}, the quartic equation of y_c is re-established using Eq. (3) to Eq. (6) and solved for updated value of y_c. By back-substituting updated y_c into Eq. (5) or (6), the value of ε_{sm} is determined (tension stiffening is now accounted for).

Fifthly, by the assumption of plane sections remain plane and linear strain distribution along the depth of cross-section, the strain values ε_c and ε_{s2} can be readily calculated and the curvature κ of SFRC beam can be computed from:

$$\kappa = \frac{\varepsilon_{s1}+\varepsilon_c}{d} = \frac{\varepsilon_c}{y_c} = \frac{\varepsilon_{s1}}{d-y_c} \tag{13}$$

Finally, the external moment M is incremented and the above five steps are repeated until the last moment increment is reached and the analysis is completed. It should be noted that throughout the crack formation stage of the analysis, the effective residual strain $f_{fr,ef}$ should be used in lieu of f_{fr} in the computation procedures.

3. APPLICATION TO SFRC BEAM SPECIMENS

The proposed analytical technique is applied to analyse the moment-curvature response of 6 SFRC beam specimens with conventional longitudinal reinforcement. Laboratory testing of the beam specimens was reported in Gribniak et al. (2012) and Kaklauskas et al. (2014). The beams had uniform rectangular section of typically 280 mm in breadth and 300 mm in depth. The beam length was 3280 mm and the span was 3000 mm. Two-point loading was applied symmetrically with shear span of 1000 mm. The tension reinforcement ratio was varied at 0.3% and 0.6%, and the compression reinforcement ratio was fixed at 0.1%. The volume fraction of steel fibres was varied among 0.5%, 1.0% and 1.5%. The structural configurations and materials properties of beams are listed in Table 1. To prevent shear failure, stirrups were provided along the shear span with shear reinforcement ratio of 0.2%. Shear reinforcement was not provided along the pure bending zone.

Table 1 Configurations and properties of beam specimens

	Fibre volume (%)	Steel ratio (%)	f'_c (MPa)	f_y (MPa)	E_s (GPa)
Beam 1	0.5	0.3	55.6	606	209
Beam 2	0.5	0.6	55.6	559	205
Beam 3	1.0	0.3	48.0	606	209
Beam 4	1.0	0.6	48.0	559	205
Beam 5	1.5	0.3	52.2	606	209
Beam 6	1.5	0.6	52.2	559	205

Note: f'_c denotes the mean cylinder strength of concrete.

The moment-curvature curves obtained from experimental testing and from the proposed analytical technique are plotted in Fig. 3. It can be seen that the experimental and analytical results are in good agreement. Comparatively, the prediction results are more accurate for Beams 2, 4 and 6 with 0.6% tension reinforcement ratio. Among these three beams, higher accuracy is obtained from Beams 2 and 4 with lower steel fibre volume (0.5% and 1.0%) than Beam 6 with 1.5% steel fibre volume. The larger error for Beams 1, 3 and 5 with lower reinforcement ratio might be attributed to the inaccuracy of Model Code in evaluating the tension stiffening in lightly reinforced flexural members.

Furthermore, the semi-empirical model proposed by Campione (2008) is employed to analyse the moment-curvature response of the beams, and the results are plotted in Fig. 3 for ease of comparison. It should be noted that the moment-curvature curve between the cracking and yielding points is assumed to be linear, and the tension stiffening effect is not considered in Campione (2008). As observed from the results, Campione's model is able to yield considerably accurate predictions for Beams 2, 4 and 6. However, Campione's model leads to significantly larger errors than the proposed analytical technique for Beams 1, 3 and 5. As a possible consequence of neglecting the tension stiffening effect, the bending moment of these beams is underestimated at given curvature.

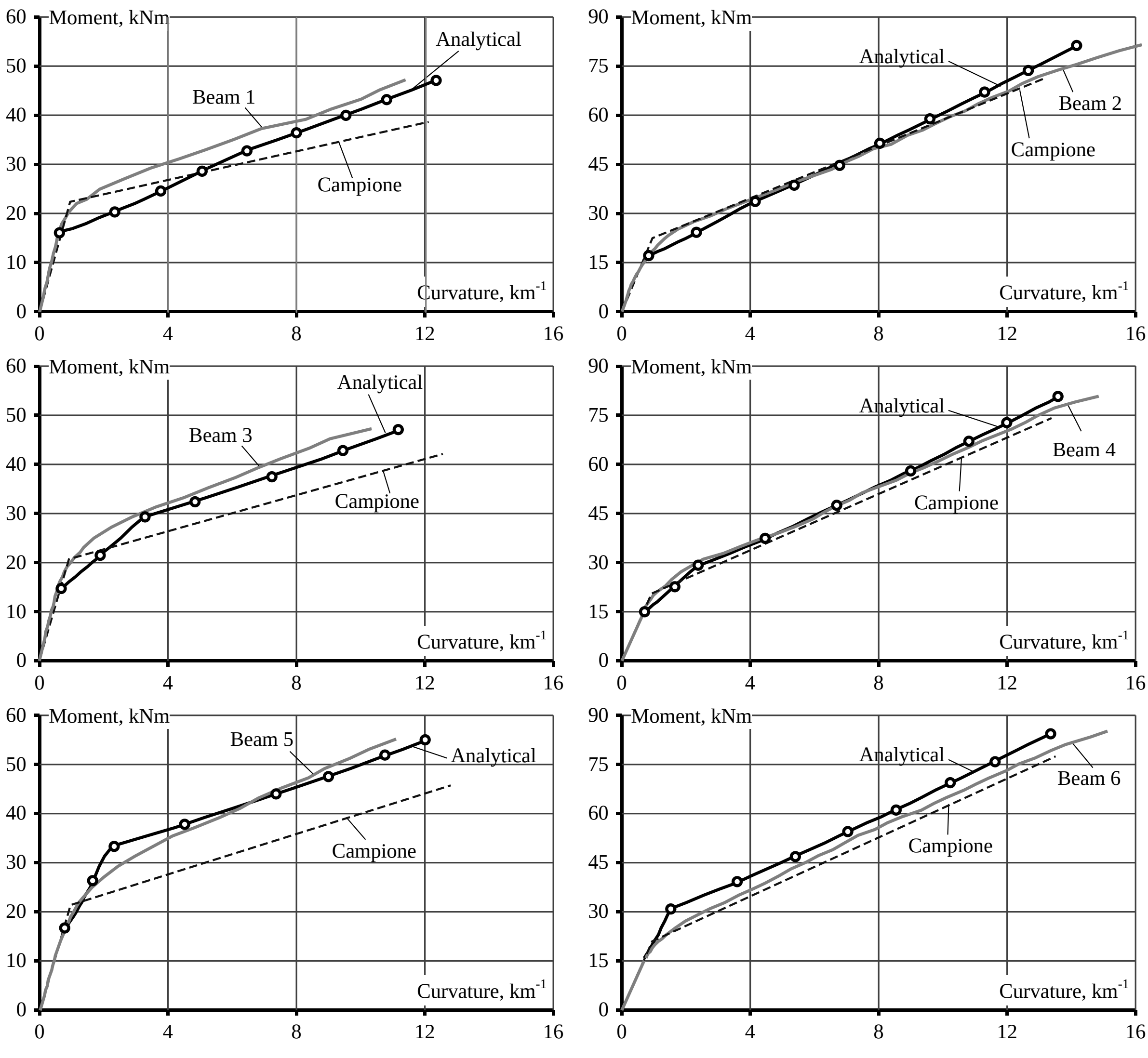

Fig. 3 Moment-curvature curves obtained from experiment and analysis

4. CONCLUSIONS

A novel analytical technique for prediction of moment-curvature response of steel fibre-reinforced concrete (SFRC) beams with conventional longitudinal reinforcement has been developed by the authors. The tension stiffening effect has been considered by means of the reinforcement-related approach. The intensity of rectangular tensile stress block is taken as the residual stress of SFRC during the stage of stable cracks, or as the newly introduced effective residual stress if the SFRC member is undergoing the crack development stage. The tension stiffening force is assessed by considering an equivalent reinforced concrete beam having the same geometrical characteristics with omission of steel fibres only. The procedures of implementing the proposed analytical technique have been explained. SFRC beam specimens with different steel fibre volumes and reinforcement ratios have been analysed to verify the validity and accuracy of the proposed technique. It has been demonstrated that the analytical moment-curvature response is in very good agreement with the experimental data. In view of the desirable results of the proposed analytical technique, it can be recommended for use in practical structural design of SFRC beams.

ACKNOWLEDGEMENTS

The study was performed under project no. 09.3.3-LMT-K-712-01-0145 that has received funding from the European Social Fund under grant agreement with the Research Council of Lithuania (LMTLT).

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