

Capturing inelastic coupling of internal forces and exact vibration frequencies for 3-d Timoshenko beam finite elements

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ABSTRACT

A three dimensional (3d) nonlinear frame finite element based on Timoshenko beam theory is presented in this analytical study by utilizing Hu-Washizu principle with displacement, strain and stress independent fields in the variational form. The finite element approximation for the beam uses shape functions for section forces that satisfy equilibrium and discontinuous section deformations along the beam. The element is free from shear-locking, and the superiority of the proposed model is displayed under nonlinear material behavior and modal analysis. Results for the inelastic coupling of 3d internal forces with proposed model are compared with results obtained from other element models and exact solutions. Vibration analyses are carried out by the use of force-based consistent mass matrix, and verification is undertaken with closed form solutions and 3d solid finite element analyses results.

1. INTRODUCTION

Development of advanced, reliable and robust finite element models for carrying out nonlinear static and vibration analyses of structures presents challenges for researchers. Frame finite element models, i.e. 1d finite element models, still provide the simplest and most efficient modeling approach in undertaking demanding solutions in structural engineering, and it appears that this trend will not likely change. The difficulty in modeling skeletal structures with solid (3d) and surface (2d) finite elements is not only caused due to increased computational demand from their use, but also due to the difficult nature of exact modeling of all types of nonlinear actions by the use of such models to a great extent. One should not forget the fact that even the use of 2d/3d solid finite element models will provide an approximate behavior to the real one. In some instances such as the modeling of the nonlinear dynamic behavior of even a low-rise steel structure with nonlinear connection response, the modeling sophistication with 3d solid models may result into the creation of a demanding 3d geometry model and

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timewise challenging analyses effort to undertake. In such instances, frame element models provide robust modeling strategies in providing a reasonable and reliable solution for the determination of the nonlinear behavior of structures. Researchers now accept the fact that frame finite element models deserve increased attention, and the popularity of some of the research oriented solution platforms, such as OpenSees, clearly demonstrate this rising trend in structural engineering scientific community.

With regards to the literature, the development of frame finite element models is basically divided into two: 1) displacement-based models and 2) force-based or mixed models. The most widely used displacement-based Timoshenko beam finite element models were proposed by (Friedman 1993) and (Reddy 1997), and this development front appears to have almost come to an end. On the other hand, the last two decades have witnessed the rise of force-based frame element models after the work by (Spacone 1996). In a later study, (Taylor 2003) demonstrated that force-based models provide accurate responses for Timoshenko beams under linear elastic material response free from shear-locking problem, and (Soydas 2013) presented an accurate inelastic 3d Timoshenko frame element based on Hu-Washizu functional that uses force interpolation functions. In a recent study by (Saritas 2015), inclusion of nonlinear and hysteretic behavior of semi-rigid connection response in steel structures was achieved without the need to introduce additional nodes and degrees of freedom in force-based element formulation. With regards to the vibration studies on force-based elements, (Soydas 2016) recently provided a force-based consistent mass matrix by using the approach of (Molins 1998), where this approach can be employed as part of their 3d frame element model based on Hu-Washizu functional. The paper in this work will combine the prior efforts by the authors of this paper and will demonstrate the superiority of the use of force-based approach not only in capturing inelastic behavior of structural members, but also in capturing vibration characteristics of structural members of various shapes without the need for specification of any displacement-field for each case.

2. FRAME ELEMENT MODEL

2.1 Kinematics of the element

Timoshenko beam theory can be extended for a 3d geometry by the use of following equation:

$$\begin{Bmatrix} u_x(x, y, z) \\ u_y(x, y, z) \\ u_z(x, y, z) \end{Bmatrix} = \begin{Bmatrix} u(x) - y\theta_z(x) + z\theta_y(x) \\ v(x) - z\theta_x(x) \\ w(x) + y\theta_x(x) \end{Bmatrix} \quad (1)$$

where the terms on the left side are the displacements in x , y and z directions on any point in the section; and the definitions for other terms can be deduced from beam theory. The non-zero strain components are the normal strain in the x direction and

shear strains on the plane that is normal to x , i.e. xy and xz components of shear strain. These are calculated from section deformations as follows;

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} u'(x) - y\theta'_z(x) + z\theta'_y(x) \\ -\theta'_z(x) + v'(x) - z\theta'_x(x) \\ \theta'_y(x) + w'(x) + y\theta'_x(x) \end{Bmatrix} = \begin{Bmatrix} \varepsilon_a(x) - y\kappa'_z(x) + z\kappa'_y(x) \\ \gamma_y(x) - z\varphi(x) \\ \gamma_z(x) + y\varphi(x) \end{Bmatrix} = \mathbf{a}_s(y, z) \mathbf{e}(x) \quad (2)$$

where $\mathbf{e}(x)$ is the section deformation vector given as;

$$\mathbf{e}(x) = [\varepsilon_a(x) \quad \gamma_y(x) \quad \gamma_z(x) \quad \varphi(x) \quad \kappa_y(x) \quad \kappa_z(x)]^T \quad (3)$$

In above $\varepsilon_a(x)$ is the axial deformation of the reference axis, $\gamma_y(x)$ and $\gamma_z(x)$ are the shear deformations about y and z axes, respectively, and $\kappa_y(x)$ and $\kappa_z(x)$ are the curvature about y and z , respectively, and section compatibility matrix $\mathbf{a}_s(y, z)$ introduced in Equation (2) is given as;

$$\mathbf{a}_s(y, z) = \begin{bmatrix} 1 & 0 & 0 & 0 & z & -y \\ 0 & 1 & 0 & -z & 0 & 0 \\ 0 & 0 & 1 & y & 0 & 0 \end{bmatrix} \quad (4)$$

2.2 Force interpolation functions

Element response is formulated in the cantilever basic system, where rigid body modes of displacements are absent in this configuration, and force interpolation functions can be more easily attained. It is assumed that the member is fixed at left node and free at right node. In order to define the parameters at fixed and free ends of the element, subscripts "0" and "L" are used hereafter, respectively. Basic element forces at free end are denoted with \mathbf{q} , and these can be related to internal section forces $\mathbf{s}(x)$ by using the force interpolation matrix for the cantilever beam configuration $\mathbf{b}(x, L)$ as follows;

$$\mathbf{s}(x) = \mathbf{b}(x, L) \mathbf{q} \quad (5)$$

$$\mathbf{s}(x) = [N(x) \quad V_y(x) \quad V_z(x) \quad T(x) \quad M_y(x) \quad M_z(x)]^T \quad (6)$$

$$\mathbf{b}(x, L) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & (x-L) & 0 & 1 & 0 \\ 0 & (L-x) & 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

By the use of Equation (5), it is possible to attain exact equilibrium between the forces at free end of the element and forces at any section that is x units away from the fixed end. Section forces are axial force $N(x)$, shear forces in y and z -directions ($V_y(x)$ and $V_z(x)$), torsion about longitudinal axis $T(x)$, and moments about y and z -axes ($M_y(x)$ and $M_z(x)$), respectively as given in Equation (6).

2.2 Finite Element Formulation

Variational form of the element is written by considering independent element nodal displacements \mathbf{u} , element basic forces \mathbf{q} , and section deformations \mathbf{e} . Extension to dynamic case is achieved through introduction of inertial forces $\mathbf{m}\ddot{\mathbf{u}}$ acting at nodes by considering D'Alembert's principle to get the following variational form of the element

$$\begin{aligned} \delta\Pi_{\text{HW}} = & \int_0^L \delta\mathbf{e}^T (\hat{\mathbf{s}}(\mathbf{e}(x)) - \mathbf{b}(x, L)\mathbf{q}) dx - \delta\mathbf{q}^T \int_0^L \mathbf{b}^T(x, L)\mathbf{e}(x) dx + \delta\mathbf{q}^T \mathbf{a}_g \mathbf{u} \\ & + \delta\mathbf{u}^T \mathbf{a}_g^T \mathbf{q} + \delta\mathbf{u}^T \mathbf{m}\ddot{\mathbf{u}} - \delta\mathbf{u}^T \mathbf{p}_{\text{app}} = 0 \end{aligned} \quad (8)$$

Above equation can also be obtained by considering the general Hu-Washizu variational form with extension to dynamic case by (Barr 1966). Equation (8) should hold for arbitrary $\delta\mathbf{u}$, $\delta\mathbf{q}$ and $\delta\mathbf{e}$, thus the following three equations should be satisfied in order for the Hu-Washizu variational to be zero.

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{p} = \mathbf{p}_{\text{app}}; \quad \text{where } \mathbf{p} = \mathbf{a}_g^T \mathbf{q} \quad (9)$$

$$\mathbf{v} = \int_0^L \mathbf{b}^T(x, L)\mathbf{e}(x) dx; \quad \text{where } \mathbf{v} = \mathbf{a}_g \mathbf{u} \quad (10)$$

$$\hat{\mathbf{s}}(\mathbf{e}(x)) = \mathbf{b}(x, L)\mathbf{q} \quad (11)$$

Equation (9) is the equation of motion that holds for linear or nonlinear material response, and this equation can be assembled for each element to get structure's equation of motion. A numerical time integration scheme can be employed to get a solution, and influence of viscous damping can be simply achieved by adding $\mathbf{c}\dot{\mathbf{u}}$ to the left hand side of the equation, where \mathbf{c} is the damping matrix, or it is also possible to determine resisting forces \mathbf{p} not only in terms of displacements \mathbf{u} but also as a function of velocities $\dot{\mathbf{u}}$.

Equations (10) and (11) are related to the element state determination, i.e. these equations can be solved independent of Equation (9), and then the solution can be condensed out into Equation (9) such that the equations of motion can be assembled for all elements.. In general, state determination of the element requires an iterative solution in the case of nonlinear behavior.

It is possible to relate section forces given in Equation (11) to section deformations given in Equation (3) as follows;

$$\hat{\mathbf{s}} = \mathbf{s}(\mathbf{e}) \quad \text{and} \quad \mathbf{k}_s = \frac{\partial \hat{\mathbf{s}}}{\partial \mathbf{e}} = \int_A \mathbf{a}_s^T \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} \mathbf{a}_s dA \quad (12)$$

where \mathbf{k}_s is the section stiffness matrix. For linear elastic material response, section stiffness matrix is calculated as;

$$\mathbf{k}_s = \int_A \mathbf{a}_s^T(y, z) \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \mathbf{a}_s(y, z) dA \quad (13)$$

In Equation (13), E is the modulus of elasticity of a material point, G is the shear modulus. Shear correction is considered to the terms of the stiffness matrix with G in order to estimate the shear strain energy accurately for the linear elastic portion of element response.

Equation (12) can be rearranged as $\mathbf{e} = \mathbf{k}_s^{-1} \hat{\mathbf{s}}$ for a linear elastic material to obtain the section deformations from section forces. Substitution of section deformations to Equation (10) gives:

$$\mathbf{a}_g \mathbf{u} = \mathbf{v} = \mathbf{f} \mathbf{q}; \quad \text{where} \quad \mathbf{f} = \int_0^L \mathbf{b}^T(x, L) \mathbf{k}_s^{-1}(x) \mathbf{b}(x, L) dx \quad (14)$$

In above equation \mathbf{f} is the flexibility of the element in the basic system. Further substitution of above equation in Equation (9) results in

$$\mathbf{m} \ddot{\mathbf{u}} + \mathbf{k} \mathbf{u} = \mathbf{p}_{app}; \quad \text{where} \quad \mathbf{k} = \mathbf{a}_g^T \mathbf{f}^{-1} \mathbf{a}_g \quad (15)$$

where \mathbf{k} is the 12x12 element stiffness matrix in the global coordinates.

2.2 Consistent Mass Matrix

Mass matrix of the proposed element is obtained in a consistent manner with the formulation of the element. Since the proposed element does not require the use of displacement interpolation functions and relies on the force-based interpolation functions $\mathbf{b}(x, L)$ presented in Equation (7), it is necessary to derive the displacement field along the length of the beam only due to the presence of the mass of the element through a consistent way with the formulation. It is worth to emphasize that the finally obtained mass matrix calculation that is presented in the following equations contains only force interpolation functions $\mathbf{b}(x, L)$, flexibility matrix \mathbf{f} in Equation (14) and does not explicitly need any displacement interpolation function. Derivation of the displacement field due to mass only is similar to unit dummy load method, and in depth discussion and derivation of the mass matrix is available in (Soydas 2016), where initial credit is due to the effort by (Molins 1998). With this alternative derivation of consistent mass matrix, the mass matrix of any type of beam element that is uniform or non-

uniform in geometry and with homogeneous or heterogeneous material distribution can be obtained.

The derivation of the consistent mass matrix requires the use of section mass matrix, which can be written as follows:

$$\mathbf{m}_s(x) = \int_A \mathbf{a}_s^T(y, z) \rho(x, y, z) \mathbf{a}_s(y, z) dA \quad (16)$$

where $\rho = \rho(x, y, z)$ is the mass density at a material point on the beam, and section compatibility matrix \mathbf{a}_s defined before.

Mass matrix of the element is written in a 12×12 dimension in local coordinates, i.e. in the complete system with 6 degrees of freedom per node, as follows:

$$\bar{\mathbf{m}} = \begin{bmatrix} \bar{\mathbf{m}}_{00} & \bar{\mathbf{m}}_{0L} \\ \bar{\mathbf{m}}_{L0} & \bar{\mathbf{m}}_{LL} \end{bmatrix} \quad (17)$$

where the components of element mass matrix are calculated from following sub-matrices

$$\begin{aligned} \bar{\mathbf{m}}_{LL} &= \mathbf{f}^{-1} \int_0^L \mathbf{b}^T(x, L) \mathbf{k}_s^{-1}(x) \left(\int_x^L \mathbf{b}^T(x, \xi) \mathbf{m}_s(\xi) \mathbf{f}_p(\xi) \mathbf{f}^{-1} d\xi \right) dx \\ \bar{\mathbf{m}}_{L0} &= \mathbf{f}^{-1} \int_0^L \mathbf{b}^T(x, L) \mathbf{k}_s^{-1}(x) \left(\int_x^L \mathbf{b}^T(x, \xi) \mathbf{m}_s(\xi) (\mathbf{b}^T(0, \xi) - \mathbf{f}_p(\xi) \mathbf{f}^{-1} \mathbf{b}^T(0, L)) d\xi \right) dx \\ \bar{\mathbf{m}}_{0L} &= \bar{\mathbf{m}}_{L0}^T = -\mathbf{b}(0, L) \bar{\mathbf{m}}_{LL} + \int_0^L \mathbf{b}(0, x) \mathbf{m}_s(x) \mathbf{f}_p(x) \mathbf{f}^{-1} dx \\ \bar{\mathbf{m}}_{00} &= -\mathbf{b}(0, L) \bar{\mathbf{m}}_{L0} + \int_0^L \mathbf{b}(0, x) \mathbf{m}_s(x) (\mathbf{b}^T(0, x) - \mathbf{f}_p(x) \mathbf{f}^{-1} \mathbf{b}^T(0, L)) dx \end{aligned} \quad (18)$$

In above equations, each matrix has dimensions of 6×6, and the partial flexibility matrix \mathbf{f}_p is given as follows:

$$\mathbf{f}_p(x) = \int_0^x \mathbf{b}^T(\xi, x) \mathbf{k}_s^{-1}(x) \mathbf{b}(\xi, x) d\xi \quad (19)$$

3. NUMERICAL VERIFICATIONS

3.1 Assessment of inelastic behavior

Inelastic behavior with coupling of 3d section forces obtained from proposed 3d mixed formulation frame element (named as proposed MF in following figures) and

Euler-Bernoulli version of the current mixed formulation beam element (named as EB-MF) are compared with the examples presented by (Nowzartash 2004). Analysis results in that study presented a 3d lumped plasticity beam element model named as P3D2HE and the results obtained with the elements available in ABAQUS, named as B33, PIPE31, FRAME3D and ELBOW31, where the details of these models are available in (Soydas 2013). The lumped plasticity element P3D2HE uses elastic perfectly plastic behavior and assumes the presence of plastic hinges at element ends only, while the proposed frame element model can capture spread of inelasticity both on element length and section depth, and thus does not have such limitations.

For analysis, a hollow circular section with outer diameter 101.6 mm, thickness 5.74 mm, elastic modulus 200 GPa, yield strength 350 MPa and Poisson's ratio 0.3 are considered; and the following loading cases for this pipe were presented for verification purposes. In the first loading case, a vertical load P (kN) and torque $T=P$ (kN.m) in magnitude are applied to a 6m fixed-fixed beam. The load is applied to the second node which is 4-m away from node 1 and nonlinear response is investigated by monitoring node 2 as shown in Fig. 1. It is evident from the figure that proposed element model is able to properly reflect the influence of the loads applied on the member.

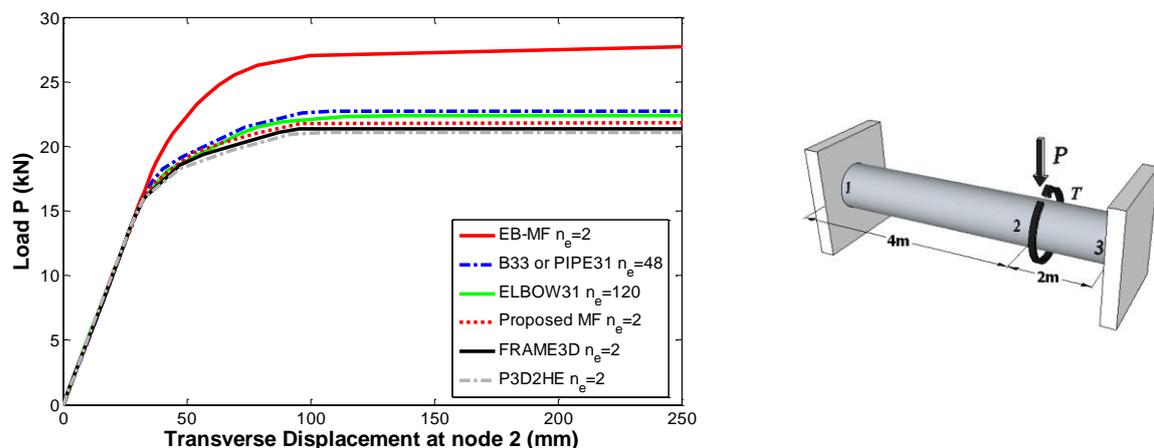


Fig. 1 Comparison of load vs. transverse displacement at node 2 for long beam.

For the second loading case, the total length of the member is decreased to 1 m and only vertical load is applied to the second node that is 0.8m from node 1 as shown in Fig. 2. In this case proposed element model appears to be the most accurate model that predicts the nonlinear behavior by capturing the theoretical value 189.5 kN-m obtained with upper bound theorem. It is interesting to observe that all other models fail to predict a reliable response in capturing the peak load in this simulation.

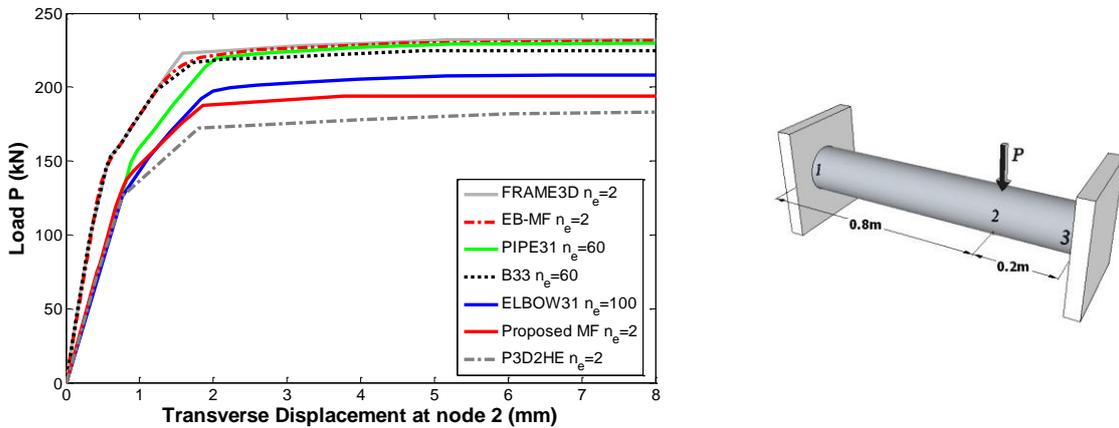


Fig. 2 Comparison of load vs. transverse displacement at node 2 for short beam.

3.2. Assessment of vibration characteristics

Vibration characteristics of the proposed element are first investigated by performing modal analyses of a linearly tapered cantilever beam with circular section. The ratio of the length of the member L to the depth of fixed end d_0 is taken as $L/d_0=3$, where $d_0=18$ units. The ratio of the diameter of the free end d_1 to the diameter of the fixed end of the member, d_1/d_0 is equal to 0.5 following analyses. Results of Abaqus are used to verify the frequency values obtained by the proposed MF element, and more detailed validation and discussion of the proposed beam element with further available studies in the literature are available in (Soydas 2016). Fig. 3 demonstrate the results of the first five frequency values obtained by the proposed MF element and Abaqus by computing ratio of frequency values $\omega_{MF}/\omega_{Abaqus}$ for given L/d_0 ratio by taking into consideration only one of the symmetrical modes in the comparisons. It is evident from the results that the proposed element is capable of capturing first five frequencies very closely by the use of $n_e = 4$, where by the way Abaqus solutions are obtained with fine mesh discretization through the use of 3d solid elements.

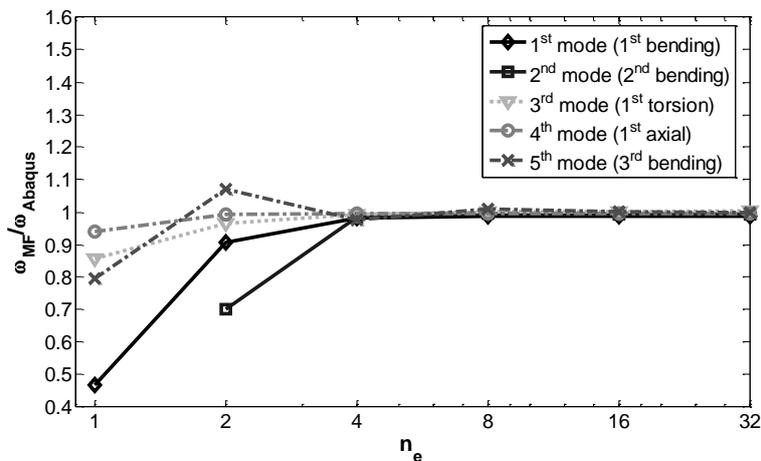
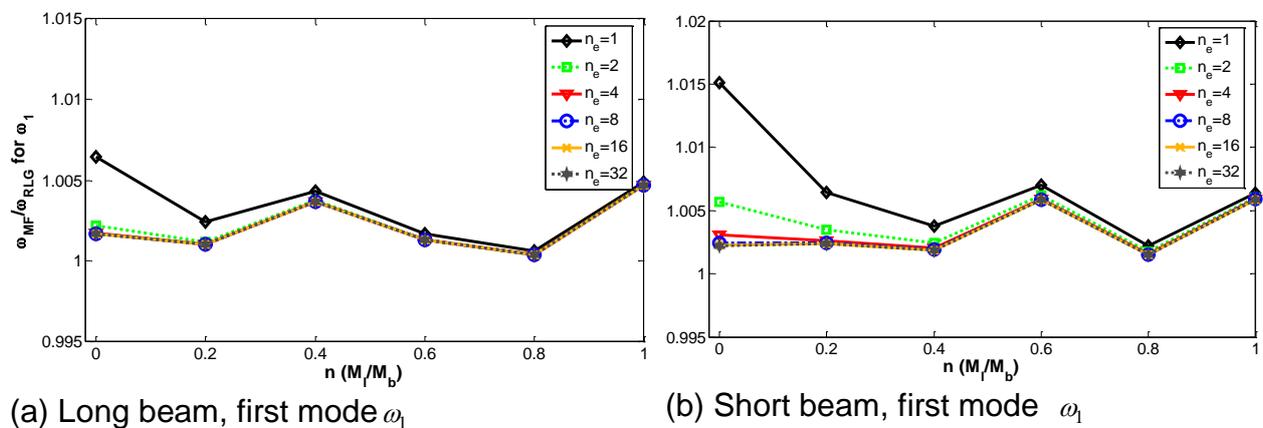


Fig. 3 Comparison of vibration frequencies for linearly tapered circular cantilever beam

In the second example, vibration frequencies for first five bending modes of a rectangular cantilever beam with lumped tip mass at the free end are obtained by the proposed element. These results are compared with the closed form solutions presented by (Rossi 1990). The width and height of the member are assumed to be 18 units in all analyses but the length of the member is varied by using the expression $r^2 = I / (AL^2)$ is equal to 10^{-7} and 10^{-2} , respectively where I is the moment of inertia and A is the area of the cross-section. $r^2 = 10^{-7}$ corresponds almost to the Euler-Bernoulli beam whereas $r^2 = 10^{-2}$ corresponds to comparatively short beam case. The last parameter that is varied in the analyses is the ratio of the mass of the lumped tip mass M_l to the mass of the beam, $n = M_l / M_b$, where $n = 0, 0.2, 0.4, 0.6, 0.8$ and 1.0 in the analyses. It should be noted that $n = 0$ means that there is no lumped mass at the free end and $n = 1.0$ stands for the case that there is a lumped tip mass that is equal to the mass of the beam ($M_l = M_b$).

The results of the analyses are summarized in Fig. 4 where the vertical axis denotes the ratio of the frequency value obtained by the proposed MF element to the frequency obtained by (Rossi 1990), ω_{MF}/ω_{RLG} . According to this figure, the proposed element can capture the first bending frequency with a difference less than 1% with single element discretization for all ranges of lumped mass for the long beam case, and less than 2% for the short beam case. The second bending frequency is captured with a difference less than 1% with $n_e = 2$ for all ranges of n for the long beam case and less than 5% for the short beam case. The fifth vibration frequency is captured with a difference less than 0.5% with $n_e = 8$ for all ranges of n for the long beam case and less than 5.1% for the short beam case. It is also evident that the fifth bending modes are almost captured exactly through the use of 8 to 16 elements per span for long beams and 16 to 32 elements per span for short beams with or without the presence of a tip mass. This last example clearly demonstrates the superiority of the proposed beam element with force-based consistent mass matrix in capturing even the higher bending modes of vibration for long and short beams with or without lumped mass present.



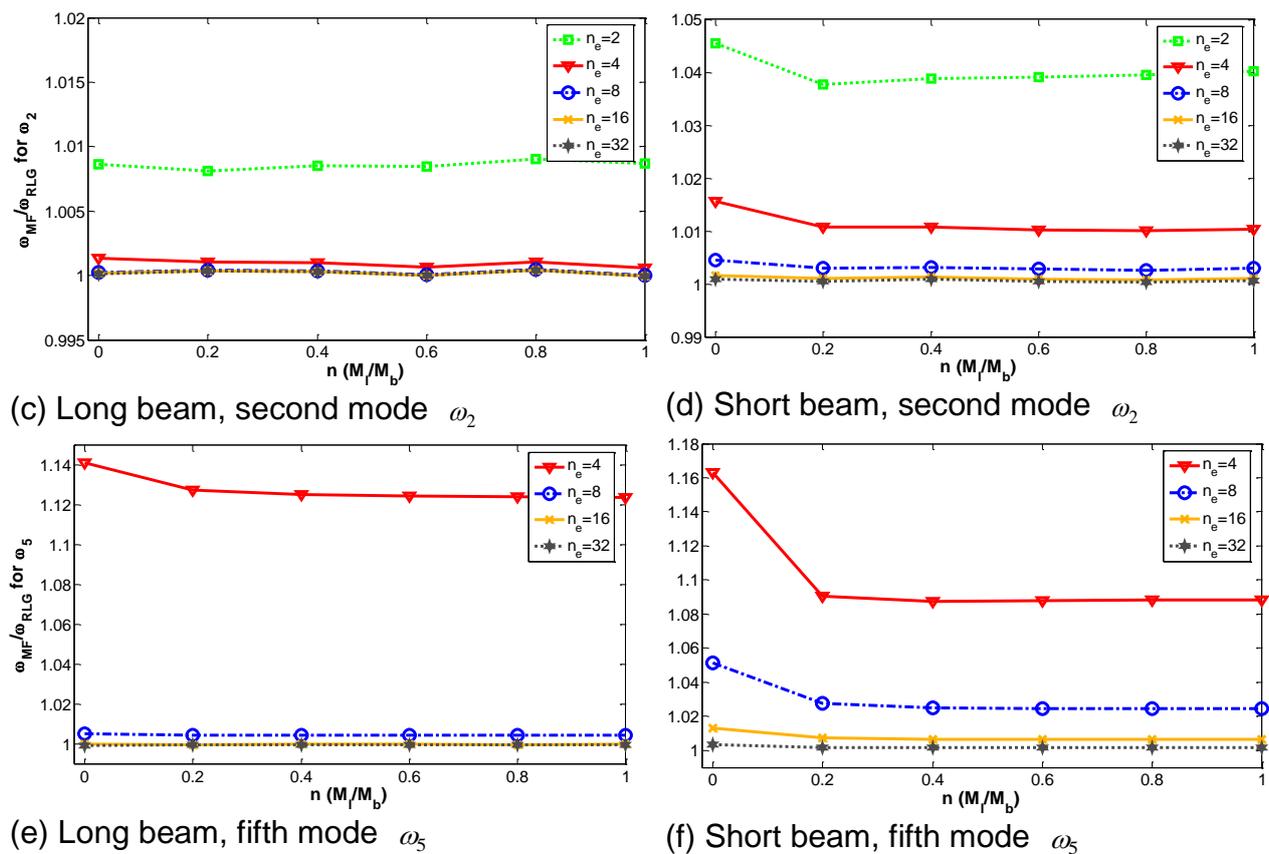


Fig. 4 Comparison of bending frequencies for a cantilever beam with tip mass

3. CONCLUSIONS

The proposed 3d Timoshenko frame element is based on three-fields Hu-Washizu functional and employs force interpolation functions. The element is free from shear-locking. Authentication and superiority of the proposed 3d element is displayed by comparing the ability of the mixed frame element to capture nonlinear coupling of axial, shear force, bending moments and torsion with the results of similar 3d models and exact solutions that are readily available in the literature. In the second part of the study, the advantage of using force-based consistent mass matrix in vibration analysis is displayed by comparing the frequency values obtained by proposed 3d mixed element with the frequency values obtained with closed form solutions available in the literature, as well as with the results of modal analysis by a finite element analysis software for members that have different uniformity and various cross sections.

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