





shear strains on the plane that is normal to  $x$ , i.e.  $xy$  and  $xz$  components of shear strain. These are calculated from section deformations as follows;

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} u'(x) - y\theta'_z(x) + z\theta'_y(x) \\ -\theta_z(x) + v'(x) - z\theta'_x(x) \\ \theta_y(x) + w'(x) + y\theta'_x(x) \end{Bmatrix} = \begin{Bmatrix} \varepsilon_a(x) - y\kappa_z(x) + z\kappa_y(x) \\ \gamma_y(x) - z\varphi(x) \\ \gamma_z(x) + y\varphi(x) \end{Bmatrix} = \mathbf{a}_s(y, z) \mathbf{e}(x) \quad (2)$$

where  $\mathbf{e}(x)$  is the section deformation vector given as;

$$\mathbf{e}(x) = [\varepsilon_a(x) \quad \gamma_y(x) \quad \gamma_z(x) \quad \varphi(x) \quad \kappa_y(x) \quad \kappa_z(x)]^T \quad (3)$$

In above  $\varepsilon_a(x)$  is the axial deformation of the reference axis,  $\gamma_y(x)$  and  $\gamma_z(x)$  are the shear deformations about  $y$  and  $z$  axes, respectively, and  $\kappa_y(x)$  and  $\kappa_z(x)$  are the curvature about  $y$  and  $z$ , respectively, and section compatibility matrix  $\mathbf{a}_s(y, z)$  introduced in Equation (2) is given as;

$$\mathbf{a}_s(y, z) = \begin{bmatrix} 1 & 0 & 0 & 0 & z & -y \\ 0 & 1 & 0 & -z & 0 & 0 \\ 0 & 0 & 1 & y & 0 & 0 \end{bmatrix} \quad (4)$$

## 2.2 Force interpolation functions

Element response is formulated in the cantilever basic system, where rigid body modes of displacements are absent in this configuration, and force interpolation functions can be more easily attained. It is assumed that the member is fixed at left node and free at right node. In order to define the parameters at fixed and free ends of the element, subscripts "0" and "L" are used hereafter, respectively. Basic element forces at free end are denoted with  $\mathbf{q}$ , and these can be related to internal section forces  $\mathbf{s}(x)$  by using the force interpolation matrix for the cantilever beam configuration  $\mathbf{b}(x, L)$  as follows;

$$\mathbf{s}(x) = \mathbf{b}(x, L) \mathbf{q} \quad (5)$$

$$\mathbf{s}(x) = [N(x) \quad V_y(x) \quad V_z(x) \quad T(x) \quad M_y(x) \quad M_z(x)]^T \quad (6)$$

$$\mathbf{b}(x, L) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & (x-L) & 0 & 1 & 0 \\ 0 & (L-x) & 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

By the use of Equation (5), it is possible to attain exact equilibrium between the forces at free end of the element and forces at any section that is  $x$  units away from the fixed end. Section forces are axial force  $N(x)$ , shear forces in  $y$  and  $z$ -directions ( $V_y(x)$  and  $V_z(x)$ ), torsion about longitudinal axis  $T(x)$ , and moments about  $y$  and  $z$ -axes ( $M_y(x)$  and  $M_z(x)$ ), respectively as given in Equation (6).

## 2.2 Finite Element Formulation

Variational form of the element is written by considering independent element nodal displacements  $\mathbf{u}$ , element basic forces  $\mathbf{q}$ , and section deformations  $\mathbf{e}$ . Extension to dynamic case is achieved through introduction of inertial forces  $\mathbf{m}\ddot{\mathbf{u}}$  acting at nodes by considering D'Alembert's principle to get the following variational form of the element

$$\begin{aligned} \delta\Pi_{\text{HW}} = & \int_0^L \delta\mathbf{e}^T (\hat{\mathbf{s}}(\mathbf{e}(x)) - \mathbf{b}(x, L)\mathbf{q}) dx - \delta\mathbf{q}^T \int_0^L \mathbf{b}^T(x, L)\mathbf{e}(x) dx + \delta\mathbf{q}^T \mathbf{a}_g \mathbf{u} \\ & + \delta\mathbf{u}^T \mathbf{a}_g^T \mathbf{q} + \delta\mathbf{u}^T \mathbf{m}\ddot{\mathbf{u}} - \delta\mathbf{u}^T \mathbf{p}_{\text{app}} = 0 \end{aligned} \quad (8)$$

Above equation can also be obtained by considering the general Hu-Washizu variational form with extension to dynamic case by (Barr 1966). Equation (8) should hold for arbitrary  $\delta\mathbf{u}$ ,  $\delta\mathbf{q}$  and  $\delta\mathbf{e}$ , thus the following three equations should be satisfied in order for the Hu-Washizu variational to be zero.

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{p} = \mathbf{p}_{\text{app}}; \quad \text{where } \mathbf{p} = \mathbf{a}_g^T \mathbf{q} \quad (9)$$

$$\mathbf{v} = \int_0^L \mathbf{b}^T(x, L)\mathbf{e}(x) dx; \quad \text{where } \mathbf{v} = \mathbf{a}_g \mathbf{u} \quad (10)$$

$$\hat{\mathbf{s}}(\mathbf{e}(x)) = \mathbf{b}(x, L)\mathbf{q} \quad (11)$$

Equation (9) is the equation of motion that holds for linear or nonlinear material response, and this equation can be assembled for each element to get structure's equation of motion. A numerical time integration scheme can be employed to get a solution, and influence of viscous damping can be simply achieved by adding  $\mathbf{c}\dot{\mathbf{u}}$  to the left hand side of the equation, where  $\mathbf{c}$  is the damping matrix, or it is also possible to determine resisting forces  $\mathbf{p}$  not only in terms of displacements  $\mathbf{u}$  but also as a function of velocities  $\dot{\mathbf{u}}$ .

Equations (10) and (11) are related to the element state determination, i.e. these equations can be solved independent of Equation (9), and then the solution can be condensed out into Equation (9) such that the equations of motion can be assembled for all elements.. In general, state determination of the element requires an iterative solution in the case of nonlinear behavior.

It is possible to relate section forces given in Equation (11) to section deformations given in Equation (3) as follows;

$$\hat{\mathbf{s}} = \mathbf{s}(\mathbf{e}) \quad \text{and} \quad \mathbf{k}_s = \frac{\partial \hat{\mathbf{s}}}{\partial \mathbf{e}} = \int_A \mathbf{a}_s^T \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} \mathbf{a}_s dA \quad (12)$$

where  $\mathbf{k}_s$  is the section stiffness matrix. For linear elastic material response, section stiffness matrix is calculated as;

$$\mathbf{k}_s = \int_A \mathbf{a}_s^T(y, z) \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \mathbf{a}_s(y, z) dA \quad (13)$$

In Equation (13),  $E$  is the modulus of elasticity of a material point,  $G$  is the shear modulus. Shear correction is considered to the terms of the stiffness matrix with  $G$  in order to estimate the shear strain energy accurately for the linear elastic portion of element response.

Equation (12) can be rearranged as  $\mathbf{e} = \mathbf{k}_s^{-1} \hat{\mathbf{s}}$  for a linear elastic material to obtain the section deformations from section forces. Substitution of section deformations to Equation (10) gives:

$$\mathbf{a}_g \mathbf{u} = \mathbf{v} = \mathbf{f} \mathbf{q}; \quad \text{where} \quad \mathbf{f} = \int_0^L \mathbf{b}^T(x, L) \mathbf{k}_s^{-1}(x) \mathbf{b}(x, L) dx \quad (14)$$

In above equation  $\mathbf{f}$  is the flexibility of the element in the basic system. Further substitution of above equation in Equation (9) results in

$$\mathbf{m} \ddot{\mathbf{u}} + \mathbf{k} \mathbf{u} = \mathbf{p}_{app}; \quad \text{where} \quad \mathbf{k} = \mathbf{a}_g^T \mathbf{f}^{-1} \mathbf{a}_g \quad (15)$$

where  $\mathbf{k}$  is the 12x12 element stiffness matrix in the global coordinates.

## 2.2 Consistent Mass Matrix

Mass matrix of the proposed element is obtained in a consistent manner with the formulation of the element. Since the proposed element does not require the use of displacement interpolation functions and relies on the force-based interpolation functions  $\mathbf{b}(x, L)$  presented in Equation (7), it is necessary to derive the displacement field along the length of the beam only due to the presence of the mass of the element through a consistent way with the formulation. It is worth to emphasize that the finally obtained mass matrix calculation that is presented in the following equations contains only force interpolation functions  $\mathbf{b}(x, L)$ , flexibility matrix  $\mathbf{f}$  in Equation (14) and does not explicitly need any displacement interpolation function. Derivation of the displacement field due to mass only is similar to unit dummy load method, and in depth discussion and derivation of the mass matrix is available in (Soydas 2016), where initial credit is due to the effort by (Molins 1998). With this alternative derivation of consistent mass matrix, the mass matrix of any type of beam element that is uniform or non-

uniform in geometry and with homogeneous or heterogeneous material distribution can be obtained.

The derivation of the consistent mass matrix requires the use of section mass matrix, which can be written as follows:

$$\mathbf{m}_s(x) = \int_A \mathbf{a}_s^T(y, z) \rho(x, y, z) \mathbf{a}_s(y, z) dA \quad (16)$$

where  $\rho = \rho(x, y, z)$  is the mass density at a material point on the beam, and section compatibility matrix  $\mathbf{a}_s$  defined before.

Mass matrix of the element is written in a 12×12 dimension in local coordinates, i.e. in the complete system with 6 degrees of freedom per node, as follows:

$$\bar{\mathbf{m}} = \begin{bmatrix} \bar{\mathbf{m}}_{00} & \bar{\mathbf{m}}_{0L} \\ \bar{\mathbf{m}}_{L0} & \bar{\mathbf{m}}_{LL} \end{bmatrix} \quad (17)$$

where the components of element mass matrix are calculated from following sub-matrices

$$\begin{aligned} \bar{\mathbf{m}}_{LL} &= \mathbf{f}^{-1} \int_0^L \mathbf{b}^T(x, L) \mathbf{k}_s^{-1}(x) \left( \int_x^L \mathbf{b}^T(x, \xi) \mathbf{m}_s(\xi) \mathbf{f}_p(\xi) \mathbf{f}^{-1} d\xi \right) dx \\ \bar{\mathbf{m}}_{L0} &= \mathbf{f}^{-1} \int_0^L \mathbf{b}^T(x, L) \mathbf{k}_s^{-1}(x) \left( \int_x^L \mathbf{b}^T(x, \xi) \mathbf{m}_s(\xi) (\mathbf{b}^T(0, \xi) - \mathbf{f}_p(\xi) \mathbf{f}^{-1} \mathbf{b}^T(0, L)) d\xi \right) dx \\ \bar{\mathbf{m}}_{0L} &= \bar{\mathbf{m}}_{L0}^T = -\mathbf{b}(0, L) \bar{\mathbf{m}}_{LL} + \int_0^L \mathbf{b}(0, x) \mathbf{m}_s(x) \mathbf{f}_p(x) \mathbf{f}^{-1} dx \\ \bar{\mathbf{m}}_{00} &= -\mathbf{b}(0, L) \bar{\mathbf{m}}_{L0} + \int_0^L \mathbf{b}(0, x) \mathbf{m}_s(x) (\mathbf{b}^T(0, x) - \mathbf{f}_p(x) \mathbf{f}^{-1} \mathbf{b}^T(0, L)) dx \end{aligned} \quad (18)$$

In above equations, each matrix has dimensions of 6×6, and the partial flexibility matrix  $\mathbf{f}_p$  is given as follows:

$$\mathbf{f}_p(x) = \int_0^x \mathbf{b}^T(\xi, x) \mathbf{k}_s^{-1}(x) \mathbf{b}(\xi, x) d\xi \quad (19)$$

### 3. NUMERICAL VERIFICATIONS

#### 3.1 Assessment of inelastic behavior

Inelastic behavior with coupling of 3d section forces obtained from proposed 3d mixed formulation frame element (named as proposed MF in following figures) and

Euler-Bernoulli version of the current mixed formulation beam element (named as EB-MF) are compared with the examples presented by (Nowzartash 2004). Analysis results in that study presented a 3d lumped plasticity beam element model named as P3D2HE and the results obtained with the elements available in ABAQUS, named as B33, PIPE31, FRAME3D and ELBOW31, where the details of these models are available in (Soydas 2013). The lumped plasticity element P3D2HE uses elastic perfectly plastic behavior and assumes the presence of plastic hinges at element ends only, while the proposed frame element model can capture spread of inelasticity both on element length and section depth, and thus does not have such limitations.

For analysis, a hollow circular section with outer diameter 101.6 mm, thickness 5.74 mm, elastic modulus 200 GPa, yield strength 350 MPa and Poisson's ratio 0.3 are considered; and the following loading cases for this pipe were presented for verification purposes. In the first loading case, a vertical load  $P$  (kN) and torque  $T=P$  (kN.m) in magnitude are applied to a 6m fixed-fixed beam. The load is applied to the second node which is 4-m away from node 1 and nonlinear response is investigated by monitoring node 2 as shown in Fig. 1. It is evident from the figure that proposed element model is able to properly reflect the influence of the loads applied on the member.

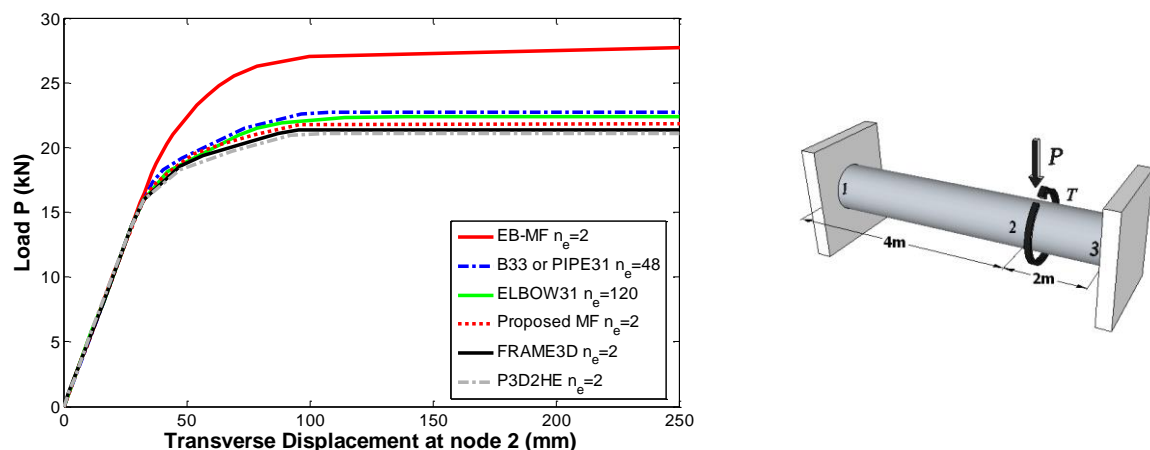


Fig. 1 Comparison of load vs. transverse displacement at node 2 for long beam.

For the second loading case, the total length of the member is decreased to 1 m and only vertical load is applied to the second node that is 0.8m from node 1 as shown in Fig. 2. In this case proposed element model appears to be the most accurate model that predicts the nonlinear behavior by capturing the theoretical value 189.5 kN-m obtained with upper bound theorem. It is interesting to observe that all other models fail to predict a reliable response in capturing the peak load in this simulation.

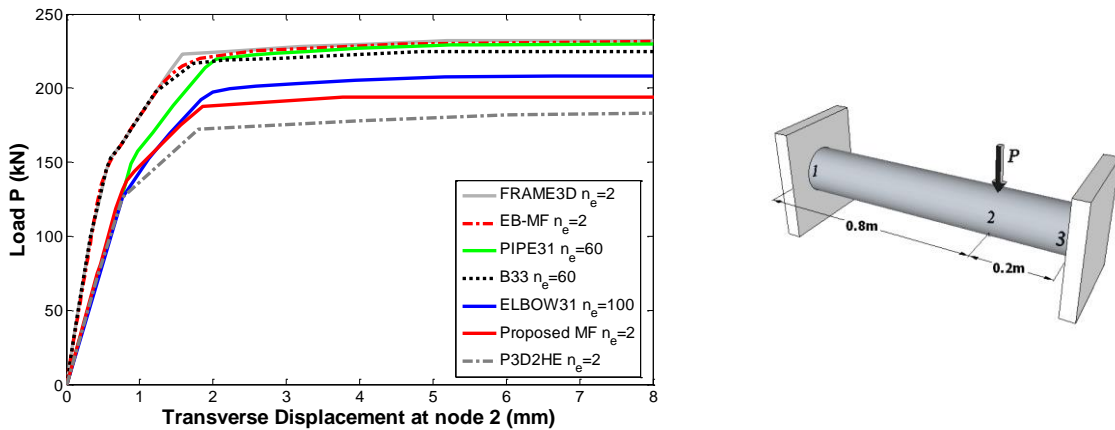


Fig. 2 Comparison of load vs. transverse displacement at node 2 for short beam.

### 3.2. Assessment of vibration characteristics

Vibration characteristics of the proposed element are first investigated by performing modal analyses of a linearly tapered cantilever beam with circular section. The ratio of the length of the member  $L$  to the depth of fixed end  $d_0$  is taken as  $L/d_0=3$ , where  $d_0=18$  units. The ratio of the diameter of the free end  $d_1$  to the diameter of the fixed end of the member,  $d_1/d_0$  is equal to 0.5 following analyses. Results of Abaqus are used to verify the frequency values obtained by the proposed MF element, and more detailed validation and discussion of the proposed beam element with further available studies in the literature are available in (Soydas 2016). Fig. 3 demonstrate the results of the first five frequency values obtained by the proposed MF element and Abaqus by computing ratio of frequency values  $\omega_{MF}/\omega_{Abaqus}$  for given  $L/d_0$  ratio by taking into consideration only one of the symmetrical modes in the comparisons. It is evident from the results that the proposed element is capable of capturing first five frequencies very closely by the use of  $n_e = 4$ , where by the way Abaqus solutions are obtained with fine mesh discretization through the use of 3d solid elements.

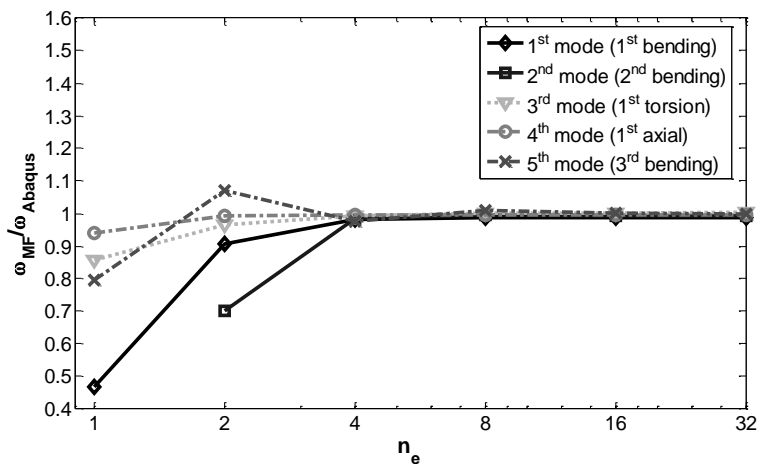


Fig. 3 Comparison of vibration frequencies for linearly tapered circular cantilever beam





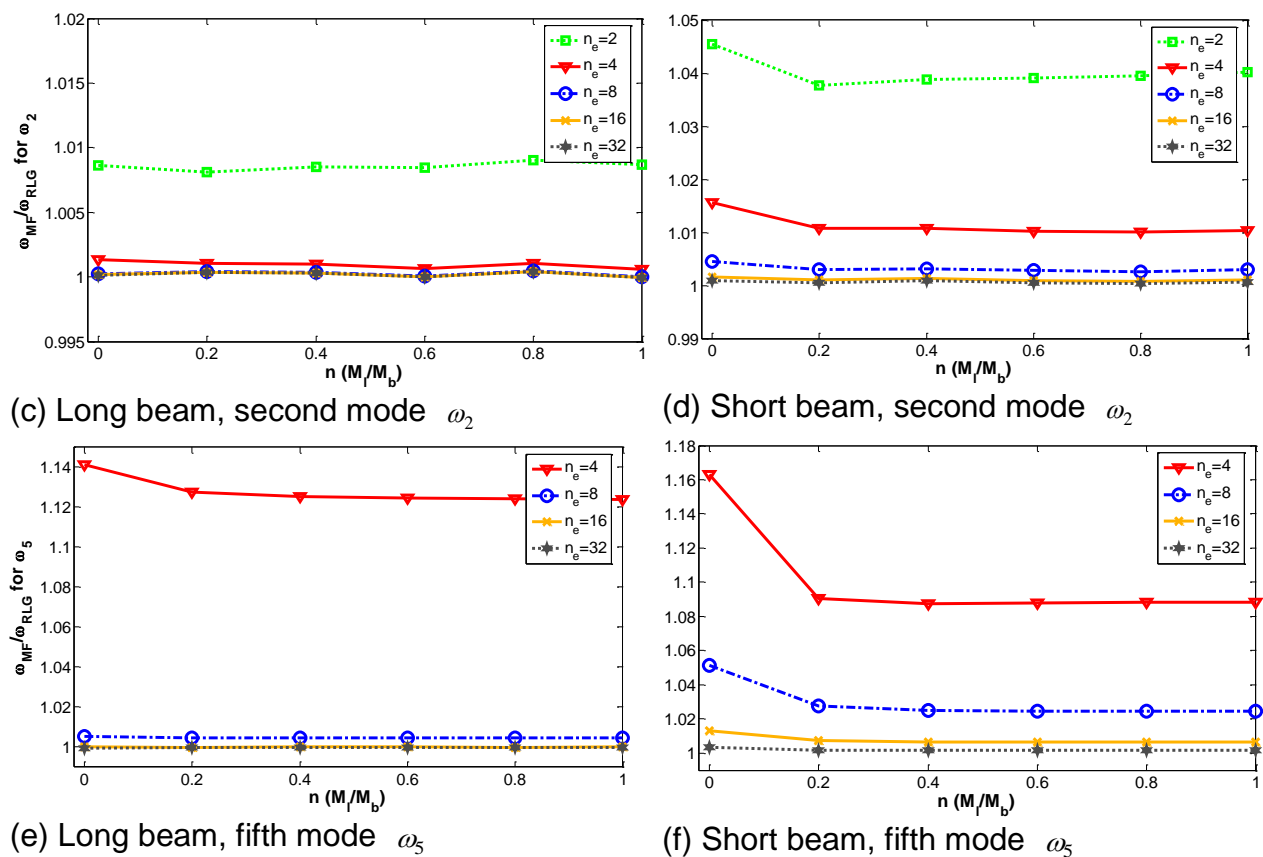


Fig. 4 Comparison of bending frequencies for a cantilever beam with tip mass

### 3. CONCLUSIONS

The proposed 3d Timoshenko frame element is based on three-fields Hu-Washizu functional and employs force interpolation functions. The element is free from shear-locking. Authentication and superiority of the proposed 3d element is displayed by comparing the ability of the mixed frame element to capture nonlinear coupling of axial, shear force, bending moments and torsion with the results of similar 3d models and exact solutions that are readily available in the literature. In the second part of the study, the advantage of using force-based consistent mass matrix in vibration analysis is displayed by comparing the frequency values obtained by proposed 3d mixed element with the frequency values obtained with closed form solutions available in the literature, as well as with the results of modal analysis by a finite element analysis software for members that have different uniformity and various cross sections.

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