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Elastic instability of non-prismatic Timoshenko beams by the power series method

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ABSTRACT

This paper presents a numerical approach based on the power series method for linear stability analysis of non-prismatic Timoshenko beams subjected to a constant axial load tangential to the beam axis. The governing system of equilibrium equations are derived from principle of stationary total potential energy. For this purpose, the total potential energy is derived from the elastic strain energy and the potential energy due to effects of the initial stresses resultants. Then the equilibrium equations lead to a unique homogeneous second-order differential equation in term of bending rotation, since in the presence of flexural and shear rigidities of cross-section of the considered Timoshenko beam, the obtained system of stability equations are coupled and simultaneous. In the case of non-uniform members, all stiffness coefficients are variable along the beam's length. The power series approximation is then adopted to ease the solution of the differential equation with variable coefficient. Finally, the critical buckling loads are determined by solving the eigenvalue problem of the obtained algebraic system. In order to illustrate the correctness and performance of proposed numerical method, one comprehensive example of Timoshenko beam with non-uniform section is presented. The obtained results are compared with available numerical or analytical solutions. The accuracy of the method is then remarked.

KEYWORDS: Non-prismatic Timoshenko beam; linear stability analysis; Power series method; Eigenvalue problem

1. INTRODUCTION

Due to improvements in fabrication process, efficiency and reduction in weight and

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cost, members with variable cross-section are extensively spread in steel structures as beams and columns. Researchers usually use Euler-Bernoulli beam theory for investing the beams with uniform or non-uniform cross-sections in which the influence of shear deformation is ignorable. They applied Timoshenko beam for static and dynamic analyses of elastic members such as towers, thick and ambulatory beams, in those the influences of shear deformation and rotary inertia are considerable and can't be ignored in the calculation process. Among the first investigations on this topic, the most important one is the study of Irie (1980), which used the transfer matrix approach for the vibration and stability analyses of Timoshenko beams subjected to a tangential follower force. Esmailzadeh (2000) studied the free vibration and stability analyses of non-prismatic Timoshenko beams subjected to axial and tangential loads. The obtained fourth order differential equations with variable coefficients are then solved by using Frobenius method. In order to investigate the static, dynamic and buckling behaviors of partial interaction composite members, Xu (2007) adopted an analytical method to solve governing differential equation. The stability analysis of Euler-Bernoulli and Timoshenko beams were solved using finite element method by Wieckowski (2007). The exponential stability of non-uniform Timoshenko beam with one internal control was investigated by Soufyane (2009). The elastic behavior of Timoshenko beams resting on nonlinear compressional and frictional Winkler foundation was investigated by Al-Azzawi (2011) using finite difference method. In the current study, in order to calculate the critical buckling loads of non-prismatic Timoshenko beams, power series method is used to solve the governing stability equation of a non-uniform Timoshenko beam.

2. EQUILIBRIUM EQUATIONS OF NON-PRISMATIC TIMOSHENKO BEAMS

A non-prismatic Timoshenko beam of Length ***L*** subjected to a constant axial load is considered in this study (Fig. 1). In the present work, it is assumed that the geometrical properties of beam cross-section vary arbitrary while the material is remained constant along member's length. For stability analysis, the beam is related to a constant axial load tangential to the axis of member. Based on the Timoshenko beam theory, the longitudinal and transverse displacement components can be respectively expressed as:

$$U(x,y,z)=u_0(x)-z\theta(x) \qquad ,W(x,y,z)=w(x) \tag{1}$$

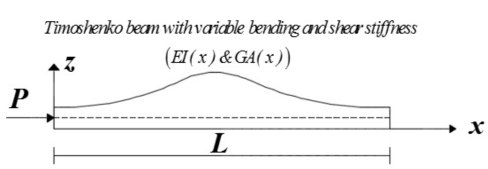

Fig. 1 A non-uniform Timoshenko beam subjected to a constant axial load at its end

In which *U* denotes the axial displacement. The displacement components *W* and θ represent vertical displacements (in z direction) and the angle of rotation of the cross-section due to bending. The equilibrium equations are derived from variation of total potential energy which is:

$$\delta\Pi = \delta\left(U_l + U_0\right) = 0 \tag{2}$$

δ illustrates a virtual variation in the last formulation. U_l and U_0 are the elastic strain energy and the potential energy due to effects of the initial stresses. Their relationships are developed as:

$$\begin{aligned} U_L &= \frac{1}{2}\int_L \int_A (\sigma_{xx}\varepsilon_{xx}^L + \tau_{xy}\gamma_{xy}^L + \tau_{xz}\gamma_{xz}^L)\,dAdx; \\ U_0 &= \int_0^L \int_A \left(\sigma_{xx}^0\varepsilon_{xx}^* + \tau_{xy}^0\gamma_{xy}^* + \tau_{xz}^0\gamma_{xz}^*\right) dAdx \end{aligned} \tag{3}$$

In these formulations, L and A express the element length and the cross-section area, in order. In Eq. (3), ε_{ij}^l and ε_{ij}^* denote the linear and the quadratic non-linear parts of strain, respectively. The Green's strain tensor components which incorporate the large displacements effects are given by:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial U_k}{\partial x_i}\frac{\partial U_k}{\partial x_j}\right) = \varepsilon_{ij}^l + \varepsilon_{ij}^* \quad \text{where } i,j,k = x,y,z \tag{4}$$

Using relationships (1) & (4), the linear and the non-linear parts of strain components are:

$$\begin{aligned} &\varepsilon_{xx}^l = u_0{}' - z\theta' \quad ,\gamma_{xy}^l = 2\varepsilon_{xy} = 0 \quad ,\gamma_{xz}^l = 2\varepsilon_{xz} = w' - \theta \\ &\qquad ,\varepsilon_{xx}^* = \frac{1}{2}(w')^2 \quad ,\gamma_{xy}^* = \gamma_{xz}^* = 0 \end{aligned} \tag{5}$$

In Eq. (3), σ_{xx}^0 and τ^0 signify initial normal stress and the mean value of the shear stress, in order. The general case of normal stress associated with constant axial force *P* is considered as:

$$\sigma_{xx}^0 = \frac{-P}{A} \qquad ,\tau_{xy}^0 = \tau_{xz}^0 = 0 \tag{6}$$

For the particular case of buckling stability context where the beam is initially under axial load, one considers that the shear stresses equal zero. The material is elastic homogeneous and isotropic. Denoting by *E* and *G* the elastic constants, the stress components of the beam are:

$$\sigma_{xx} = E\varepsilon_{xx}^l, \qquad \tau_{xy} = G\gamma_{xy}^l, \qquad \tau_{xz} = G\gamma_{xz}^l \tag{7}$$

Substituting the strain-displacement relations defined in Eq. (5), initial stresses (6) and elastic stress (7) into Eq. (3), and integration over the cross-section in the context of principal axes, the total potential energy of an elastic Timoshenko beam with variable cross-section are derived as:

$$\Pi = \frac{1}{2}\int_L EI\theta'^2 dx + \frac{1}{2}\int_L kGA(w'-\theta)dx - \frac{P}{2}\int_L w'^2 dx \tag{8}$$

In the last expression, ***k*** is the shear correction factor. I denotes the second moments of area. By variation on Eq. (8) with respect to u_0, w and θ, the equilibrium equations are derived as:

$$(EAu_0')' = 0 \tag{9}$$

$$(EI\theta')' + kGA(w' - \theta) = 0 \tag{10}$$

$$\left(kGA(w' - \theta)\right)' - Pw'' = 0 \tag{11}$$

The associated boundary conditions for Timoshenko beam are:

$$EAu_0' = 0 \quad \text{Or} \quad \delta(\partial u) = 0 \tag{12}$$

$$EI\theta' = 0 \quad \text{Or} \quad \delta(\partial \theta) = 0 \tag{13}$$

$$kGA(w' - \theta) - Pw' = 0 \quad \text{Or} \quad \delta(\partial w) = 0 \tag{14}$$

The equilibrium equations (10-11) are simultaneous differential equations due to the presence of displacement components (*w* and θ) while, the axial stability equation (Eq. (10)) is uncoupled to the mentioned ones. The axial equilibrium equation has no incidence on linear stability analysis of Timoshenko beam. The third equilibrium equation (12) by assuming that the boundary conditions in vertical direction can be transformed into:

$$w' = \left(\frac{kGA}{kGA - P}\right)\theta \tag{15}$$

Using the last expression and after some needed simplifications, the second equilibrium equation (10) is then uncoupled to the transverse displacement (*w*), the following differential equation is then derived only in term of the angle of rotation (θ):

$$(kGA)(EI\theta')' + P(kGA\theta - (EI\theta')') = 0 \tag{16}$$

In the following, in order to solve obtained stability equation (16), the power series expansions is applied. Referring to this method, all geometric properties and the displacement component of a beam are developed into power series form.

3. NUMERIAC APPROACH

In the case of non-prismatic beams, none of the geometric characteristics of the cross-section are constant. For this reason, all these terms are presented in power series form, as follows:

$$I(x) = \sum_{i=0}^{\infty} I_i x^i \qquad , A(x) = \sum_{i=0}^{\infty} A_i x^i \tag{17}$$

Where I_i and A_i are coefficients of power series at order *i*. In order to ease the solution of the stability equation, a dimensionless variable ($\varepsilon = x/L$) is introduced. Substituting Eq. (17) and the non-dimensional variable ε into the stability Eq. (16) lead to the following equation:

$$\begin{aligned}&(kG(\sum_{i=0}^{\infty} A_i L^i \varepsilon^i))\frac{d}{d\varepsilon}(E(\sum_{i=0}^{\infty} I_i L^i \varepsilon^i)\frac{d\theta}{d\varepsilon}) \\ &+P\left(L^2 kG(\sum_{i=0}^{\infty} A_i L^i \varepsilon^i)\theta(\varepsilon) - \frac{d}{d\varepsilon}(E(\sum_{i=0}^{\infty} I_i L^i \varepsilon^i)\frac{d\theta}{d\varepsilon})\right) = 0\end{aligned} \tag{18}$$

The general solution of the cross-section rotation angle is also presented by the following infinite power series of the form:

$$\theta(\varepsilon)=\sum_{i=0}^{\infty}a_i\varepsilon^i \tag{19}$$

Furthermore, introducing new variables:

$$I_i^*=I_iL^i\ ,\qquad A_i^*=A_iL^i \tag{20}$$

And substituting Eq. (19)-(20) into Eq. (18), the following equation is found:

$$\begin{aligned}&(kG\sum_{m=0}^{\infty}A_m^*\varepsilon^m)\frac{d}{d\varepsilon}(E(\sum_{i=0}^{\infty}I_i^*\varepsilon^i)(\sum_{j=0}^{\infty}(j+1)a_{j+1}\varepsilon^j))\\&+P\left(L^2kG(\sum_{i=0}^{\infty}A_i^*\varepsilon^i)(\sum_{j=0}^{\infty}a_j\varepsilon^j)-\frac{d}{d\varepsilon}(E(\sum_{i=0}^{\infty}I_i^*\varepsilon^i)(\sum_{j=0}^{\infty}(j+1)a_{j+1}\varepsilon^j))\right)=0\end{aligned} \tag{21}$$

By multiplying the two series in each terms of equation (21), the following expression is obtained:

$$\begin{aligned}\sum_{m=0}^{\infty}&\left(kEG\sum_{j=0}^{m}\sum_{i=0}^{j+1}I_i^*a_{j-i+2}A_{m-j}^*(j-i+2)(j+1)+PL^2kG\sum_{i=0}^{m}A_i^*a_{m-i}\right.\\&\left.-PE\sum_{i=0}^{m+1}I_i^*a_{m-i+2}(m-i+2)(m+1)\right)\varepsilon^m=0\end{aligned} \tag{22}$$

The following recurrence formula for the coefficient (a_{m+2}) are then obtained:

$$\begin{aligned}a_{m+2}=&\frac{1}{PEI_0^*(m+2)(m+1)}\left[-PE\sum_{i=0}^{m+1}I_i^*\,a_{m-i+2}(m-i+2)(m+1)\right.\\&\left.+kEG\sum_{j=0}^{m}\sum_{i=0}^{j+1}I_i^*a_{j-i+2}A_{m-j}^*(j-i+2)(j+1)+PL^2kG\sum_{i=0}^{m}A_i^*a_{m-i}\right]\qquad m=0,1,2,3,...\end{aligned} \tag{23}$$

With the recurrence formula, the fundamental solution of Eq. (21) can be obtained unambiguously in terms of the four constants (a_0,a_1) which can be determined by imposing the natural boundary conditions. The solution of Eq. (18) can be expressed in the following form:

$$\theta(\varepsilon)=a_0\theta_0(\varepsilon)+a_1\theta_1(\varepsilon) \tag{24}$$

Where $\theta_i\ (i=0,1)$ are the fundamental solutions of Eq. (16). Knowing that the first two coefficients (a_0,a_1) are functions of the displacements of degree of freedom (DOF). Then all the coefficients $a_i(i=2,3,4,..)$ are also functions of the displacements of DOF. The calculation procedure is done with the aid of MATLAB software.

4. RESULTS AND DISCUSSIONS

In order to investigate the accuracy and the efficiency of the power series method in stability analysis of non-uniform Timoshenko beam with arbitrary boundary conditions, a comprehensive example composed of two different parts is presented. Therefore, the linear buckling loads of Timoshenko beams with uniform and non-uniform cross-sections are carried out in the presence of rectangular cross-section. In all presented numerical examples, the modulus of elasticity, Poisson’s ratio and the shear correction factor are assumed $200GPa$, 0.3 and 5/6, in order. In the following equations, the geometric parameters at the left and right supports of the beam are indicated by the subscript 0 and 1, respectively. In order to simplify the solution procedure and the illustration of obtained results, some new non-dimensional parameters are adopted as:

$$m = (\frac{b_0}{L})^2; \qquad \lambda_{cr} = \frac{P_{cr}L^2}{EI_0} \tag{25}$$

In the following numerical cases, *m* is equal to 0.12. The aim of the first section is to define the required number of terms in power series expansions to obtain an acceptable accuracy on critical elastic buckling loads. Therefore, Table 1 gives the lowest value of elastic buckling loads of prismatic beams with different boundary conditions. Effect of the number of power series terms (*n*) considered in the proposed numerical technique on convergence is also displayed in Table 1. The obtained results by the presented technique have been compared with those obtained by the finite element solution based on combination of energy approach and basic displacement functions (BDFs) reported by Shahba (2011) and with the exact ones proposed by Wang (2004).

Table 1: Critical load parameter (λ_{cr}) comparison of power series method and other available results for beam

Number of terms of power series		
	Clamped-Free	Hinged-Hinged
	λ_{cr}	λ_{cr}
6	2.3499	-
8	2.2886	7.3149
10	2.2911	7.5720
12	2.2910	7.5443
14	2.2910	7.5460
16	2.2910	7.5459
18	2.2910	7.5459
20	2.2910	7.5459
Wang (2004)	2.2910	7.5459
Shahba (2011)	2.2911	7.5472

It should be noted that taking more than 14 terms of power series expansions for high accurate solutions involved in the stability analysis is not required.

The second section deals with the stability analysis of two sets of non-prismatic Timoshenko beams subjected to a concentrated axial load tangential to the beam axis. In all considered beams, the geometrical properties of the left end section of the member are constant. In the first case, the width of beam's section (b_0) at the left support is made to diminish to (b_1) at the other end with a linear variation, while its length is remained constant. In **Case B**, the length (*h*) and width (*b*) of considered rectangle are simultaneously allowed to vary linearly along the length with same tapering ratio $\beta = h_1 / h_0 = b_1 / b_0$. In this example, the tapering parameter can change from $\beta = 0$ (prismatic beam) to $\beta = 0.2, 0.4, 0.6, 0.8$ (non-uniform ones). For both cases, the moment of inertia and the cross-section area can be represented in the following forms:

Case A: $I(\varepsilon) = I_0(1 - \beta\varepsilon)^3; A(\varepsilon) = A_0(1 - \beta\varepsilon)$ and **Case B**: $I(\varepsilon) = I_0(1 - \beta\varepsilon)^4; A(\varepsilon) = A_0(1 - \beta\varepsilon)^2$ (26)

Table 2: Critical axial load parameter (λ_{cr}) for different non-prismatic beams

Tapering ratio β	**Case A**		**Case B**	
	Clamped-Free	Hinged-Hinged	Clamped-Free	Hinged-Hinged
	λ_{cr}	λ_{cr}	λ_{cr}	λ_{cr}
0	2.2910	7.5459	2.2910	7.5459
0.2	1.8839	5.6742	1.7421	5.0354
0.4	1.4653	3.9124	1.2034	2.9236
0.6	1.0261	2.2826	0.6819	1.2818
0.8	0.6572	1.2085	0.3900	0.6664

CONCLUSIONS

In this paper, the stability analysis of non-uniform Timoshenko beams was studied using numerical method based on power series expansions. The results obtained using aforementioned method are in close agreement to other benchmark solutions available in the literature. In most cases, the buckling loads of non-uniform members can be determined with a very good accuracy, within a relative error, less than 1%., by considering 12 to 20 terms of power series.

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