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Free Vibration Analysis of Beams with Variable Flexural Rigidity Resting on one or two Parameter Elastic Foundations using Finite Difference Method

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ABSTRACT

In this paper, the Finite Difference Method (FDM) is applied to evaluate natural frequencies of non-prismatic beams, with different boundary conditions and resting on variable one or two parameter elastic foundations. Finite difference method is one of the most powerful numerical techniques for solving differential equations especially with variable coefficients. Between various computational methods to solve the motion equations, this method requires a minimum of computing stages and is therefore very suitable approach for engineering analysis where the exact solution is very difficult to obtain. The main idea of this method is replacing derivatives present in the free vibration equation and boundary condition equations with finite difference expressions. The natural frequencies are determined by solving the eigenvalue problem of the obtained algebraic system resulting from FDM expansions. In order to illustrate the correctness and performance of the method, a comprehensive numerical example of non-uniform beams is presented. The results are compared with the finite element results using Ansys software and other available numerical and analytical solutions.

KEYWORDS: Non-uniform beams; variable elastic foundation; Free vibration analysis; Finite difference method; Eigenvalue problem

1. INTRODUCTION

The investigations of elastic buckling load and natural frequency of beam-columns resting on elastic foundations is one of the important points in the design of many structures related to soil-structure interaction (the foundation of buildings, pipelines embedded in soil, highway pavements, etc.). Researchers adopted different numerical techniques such as finite element method and power series approach to study free vibration behavior of theses members. Among the first investigations on this

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topic, the most important ones are the studies of Eisenberger (1987) and Matsunaga (1999), which used power series expansions for buckling and free vibration analyses of beams on elastic foundations. Girgin (2005) derived the static and dynamic stiffness matrices based on Mohr method for non-uniform members resting on variable elastic foundations. Baki (2013) introduced an analytical solution for studying the free vibration behavior and calculating the natural frequencies of the beams with different boundary conditions on elastic foundation. Mirzabeigy (2014) presented a semi-analytical method based on differential transform method to obtain the dimensionless natural frequencies of non-uniform beams resting on an elastic foundation. In this study, natural frequencies of non-prismatic beams resting on variable two parameter elastic foundations are investigated by using Finite Difference Method (FDM).

2. MOTION EQUATION FOR BEAMS RESTING ON A TWO PARAMETER ELASTIC FOUNDATION

In this study, a non-prismatic beam of length ***L*** with variable flexural rigidity ***EI(x)*** resting on two-parameter elastic foundations is considered (Fig. 1a). The motion differential equation of non-uniform beams resting on variable elastic foundations can be expressed as follow:

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w}{dx^2}\right)-\frac{d}{dx}\left(K_1(x)\frac{dw}{dx}\right)+\left(K(x)-m(x)\omega^2\right)w(x)=0 \tag{1}$$

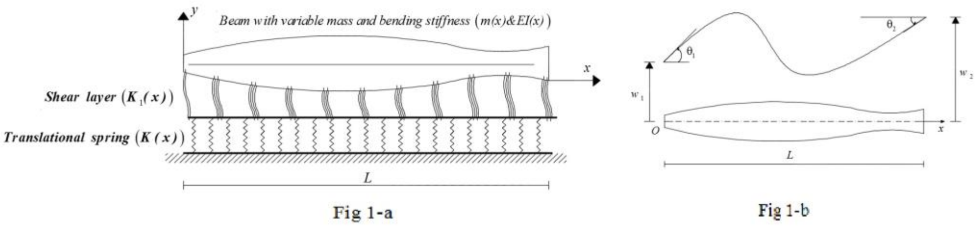


Fig. 1 Non-prismatic beam resting on a two-parameter elastic foundation (a) and Degrees of freedom for a column element (b)

Pondering to Fig. 1b, two degrees of freedom exist at each node of elements in plane bending; vertical displacement (w_1,w_2) and rotation (θ_1,θ_2). Therefore, for each end of a beam depending on its condition, two boundary conditions can be considered as follows:

Free end: $\frac{d^2w}{dx^2}=0$ and $\frac{d^3w}{dx^3}+\frac{K_1(x)}{EI}\cdot\frac{dw}{dx}=0$ (2)

Pinned support: $w=0$ and $\frac{d^2w}{dx^2}=0$ (3)

Clamped support: $w=0$ and $\frac{dw}{dx}=0$ (4)

3. FDM FORMULATION OF THE PROBLEM

For solving differential equations with generalized end conditions, finite difference method is supposed to be a dominant numerical method. Finite difference approach is a numerical iterative procedure that involves the use of successive approximation to obtain solutions of differential equations especially with variable coefficients. This numerical method is based on replacing each term of derivatives present in the differential equation and its related boundary conditions with finite difference formulations. The basis of this method is to approximate the function of derivatives with Taylor series expansions.
In order to apply the finite difference method to the motion equation (1), the beam member with length of L is assumed to be sub-divided into n parts, each of which equals to the length $h=L/n$, as shown in Fig. 2. Therefore, there are n+1 nodes along the beam's length whose numbering starts with 0 at the left end finishes to n at the other side.

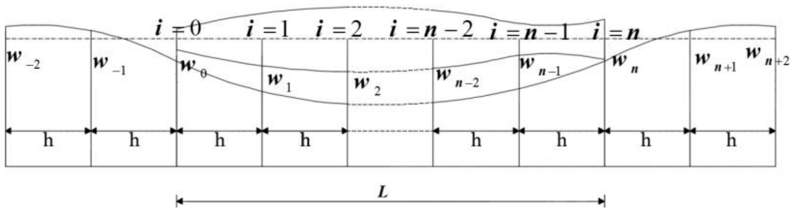


Fig. 2 equally spaced grid point along the column's length in finite difference method

According to central finite difference method and in the presence of first to fourth order derivatives of vertical displacement of the considered element, derivatives of displacement for a discrete column are formulated as follows:

$$\frac{dw}{dx}=\frac{w_{i+h}-w_{i-h}}{2h} \tag{5a}$$

$$\frac{d^2w}{dx^2}=\frac{w_{i+h}-2w_i+w_{i-h}}{h^2} \tag{5b}$$

$$\frac{d^3w}{dx^3}=\frac{w_{i+2h}-2w_{i+h}+2w_{i-h}-w_{i-2h}}{2h^3} \tag{5c}$$

$$\frac{d^4w}{dx^4}=\frac{w_{i+2h}-4w_{i+h}+6w_i-4w_{i-h}+w_{i-2h}}{h^4} \tag{5d}$$

In which: w_{i-2h} ، w_{i-h} ، w_i ، w_{i+h} and w_{i+2h} are displacements of column in five points with the space of h. By substituting relations (5a) to (5d) in the extension form of equation (1), and after simplification, the governing differential equation in finite difference form at node i, can be expressed as follow:

$$\begin{aligned}&w_{i+2h}\left[EI(x)+h\frac{dEI(x)}{dx}\right]\\&+w_{i+h}\left[-4EI(x)-2h\frac{dEI(x)}{dx}+h^2\frac{d^2EI(x)}{dx^2}-h^2K_1(x)-0.5h^3\frac{dK_1(x)}{dx}\right]+\\&w_i\left[6EI(x)-2h^2\frac{d^2EI(x)}{dx^2}+2h^2K_1(x)+h^4K(x)-m(x)h^4\omega^2\right]\\&+w_{i-h}\left[-4EI(x)+2h\frac{dEI(x)}{dx}+h^2\frac{d^2EI(x)}{dx^2}-h^2K_1(x)+0.5h^3\frac{dK_1(x)}{dx}\right]\\&+w_{i-2h}\left[EI(x)-h\frac{dEI(x)}{dx}\right]=0\end{aligned} \tag{6}$$

Furthermore, introducing five new parameters:

$$A(x)=EI(x)+h\frac{dEI(x)}{dx} \tag{7a}$$

$$B(x)=-4EI(x)-2h\frac{dEI(x)}{dx}+h^2\frac{d^2EI(x)}{dx^2}+h^2K_1(x)-0.5h^3\frac{dK_1(x)}{dx} \tag{7b}$$

$$C(x)=6EI(x)-2h^2\frac{d^2EI(x)}{dx^2}-2h^2K_1(x)+h^4K(x)-m(x)h^4\omega^2 \tag{7c}$$

$$D(x)=-4EI(x)+2h\frac{dEI(x)}{dx}+h^2\frac{d^2EI(x)}{dx^2}+h^2K_1(x)+0.5h^3\frac{dK_1(x)}{dx} \tag{7d}$$

$$E(x)=EI(x)-h\frac{dEI(x)}{dx} \tag{7e}$$

And substituting Eq. (7a)-(7e) into Eq. (6), the following expression is found:

$$w_{i+2h}\left[A(x)\right]+w_{i+h}\left[B(x)\right]+w_i\left[C(x)\right]+w_{i-h}\left[D(x)\right]+w_{i-2h}\left[E(x)\right]=0 \tag{8}$$

Last expression should be written for n+1 grid points of a divided element; thus, n+1 equations are derived including n+5 unknown parameters ($w_{-2}, w_{-1}, w_0, w_1, ..., w_n, w_{n+1}, w_{n+2}$). In order to solve the system of equation obtained based on central finite difference method, four extra equations eventuated from boundary conditions of the beam are required. According to finite difference formulations, the introduced boundary conditions in equations (2) to (4) can be modified for the first and final points of divisions (*i=0* and *i=n*) as follows:

Free end:

$$i=0 \quad\rightarrow\quad \begin{cases} w_1-2w_0+w_{-1}=0 \\ w_2+(-2+\frac{L^2}{n^2}\cdot\frac{K_1(x=0)}{EI(x=0)})w_1+(2-\frac{L^2}{n^2}\cdot\frac{K_1(x=0)}{EI(x=0)})w_{-1}-w_{-2}=0 \end{cases}$$

$$i=n \quad\rightarrow\quad \begin{cases} w_{n+1}-2w_n+w_{n-1}=0 \\ w_{n+2}+(-2+\frac{L^2}{n^2}\cdot\frac{K_1(x=L)}{EI(x=L)})w_{n+1}+(2-\frac{L^2}{n^2}\cdot\frac{K_1(x=L)}{EI(x=L)})w_{n-1}-w_{n-2}=0 \end{cases} \tag{9}$$

Pinned support:

$$i=0 \quad\rightarrow\quad \begin{cases} w_0=0 \\ w_1-2w_0+w_{-1}=0 \end{cases}$$

$$i=n \quad\rightarrow\quad \begin{cases} w_n=0 \\ w_{n+1}-2w_n+w_{n-1}=0 \end{cases} \tag{10}$$

Clamped support:
$$i=0 \quad \rightarrow \quad \begin{cases} w_0 = 0 \\ w_1 - w_{-1} = 0 \end{cases}$$
$$i=n \quad \rightarrow \quad \begin{cases} w_n = 0 \\ w_{n+1} - w_{n-1} = 0 \end{cases} \tag{11}$$

Therefore, finite difference approach in the presence of n equal segments along the considered member constitutes a system of simultaneous equations consisting n+5 linear equations. In the following, the simplified motion equation through FD formulation is written for each grid point without considering the corresponding equations of boundary conditions:

$$i=0 \quad \rightarrow \quad w_2\left[A(x=0)\right]+w_1\left[B(x=0)\right]+w_0\left[C(x=0)\right]+w_{-1}\left[D(x=0)\right]+w_{-2}\left[E(x=0)\right]=0$$
$$i=1 \quad \rightarrow \quad w_3\left[A(x=\frac{1\times L}{n})\right]+w_2\left[B(x=\frac{1\times L}{n})\right]+w_1\left[C(x=\frac{1\times L}{n})\right]+w_0\left[D(x=\frac{1\times L}{n})\right]+w_{-1}\left[E(x=\frac{1\times L}{n})\right]=0$$
$$i=2 \quad \rightarrow \quad w_4\left[A(x=\frac{2\times L}{n})\right]+w_3\left[B(x=\frac{2\times L}{n})\right]+w_2\left[C(x=\frac{2\times L}{n})\right]+w_1\left[D(x=\frac{2\times L}{n})\right]+w_0\left[E(x=\frac{2\times L}{n})\right]=0 \tag{12}$$
.
.
.
$$i=n \quad \rightarrow \quad w_{n+2}\left[A(x=\frac{n\times L}{n})\right]+w_{n+1}\left[B(x=\frac{n\times L}{n})\right]+w_n\left[C(x=\frac{n\times L}{n})\right]+w_{n-1}\left[D(x=\frac{n\times L}{n})\right]+w_{n-2}\left[E(x=\frac{n\times L}{n})\right]=0$$

The final equation is obtained in a matrix notation as follow:

$$\left[A\right]_{n+5\times n+5}\left\{w\right\}_{n+5\times 1}=\left\{0\right\}_{n+5\times 1} \tag{13a}$$

$$\left[A\right]=\begin{bmatrix}
a & b & c & d & 0 & 0 & . & . & . & 0 & 0 \\
e & f & g & h & 0 & 0 & . & . & . & 0 & 0 \\
E(x=0) & D(x=0) & C(x=0) & B(x=0) & A(x=0) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E(x=\frac{L}{n}) & D(x=\frac{L}{n}) & C(x=\frac{L}{n}) & B(x=\frac{L}{n}) & A(x=\frac{L}{n}) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E(x=\frac{2L}{n}) & D(x=\frac{2L}{n}) & C(x=\frac{2L}{n}) & B(x=\frac{2L}{n}) & A(x=\frac{2L}{n}) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & . & . & . & . & . & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & . & . & . & . & . & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & E(x=L) & D(x=L) & C(x=L) & B(x=L) & A(x=L) \\
0 & 0 & . & . & . & 0 & a' & b' & c' & d' & e' \\
0 & 0 & . & . & . & 0 & f' & g' & h' & i' & j'
\end{bmatrix} \tag{13.b}$$

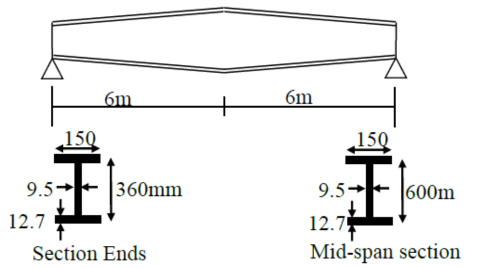
$$\left[w\right]=\left[w_{-2} \quad w_{-1} \quad w_0 \quad w_1 \quad w_2 \quad . \quad . \quad . \quad w_n \quad w_{n+1} \quad w_{n+2}\right]^{-1} \tag{13.c}$$

In which, $\left[A\right]$ and $\left\{w\right\}$ denote the expansion of coefficient matrix and displacement vector, respectively. In Eq. (13b), a, a', b, b', c, ... , i', j, j',.... are obtained based on boundary conditions. The natural frequency is then calculated by solving eigenvalue problem. The determinant of the coefficient matrix (*A*) must be zero to have non-zero answer. The smallest positive real root of the equation is considered as natural frequency. The calculation procedure is done with the aid of MATLAB software.

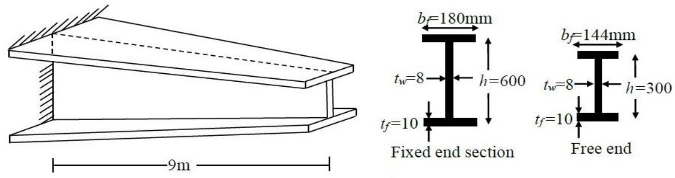
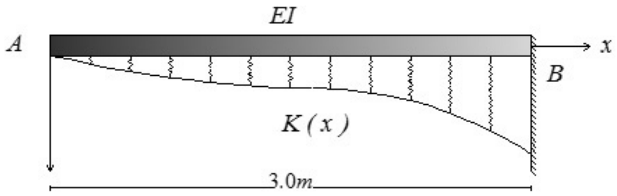
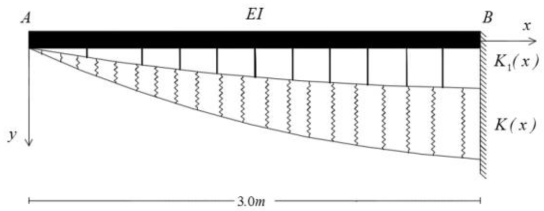
4. RESULTS AND DISCUSSIONS

In this section, in order to demonstrate the performance of finite difference method in free vibration analysis of beams resting on one or two parameter elastic foundations, a comprehensive example including four different cases is represented. The obtained results by the present solution using central finite difference method have been compared with those obtained by the finite element solutions using Ansys and other available numerical benchmarks. Table 1 gives the value of natural frequency of various types of beams for the first bending mode shape. Effect of the number of segments (*n*) considered in FDM on convergence is also displayed in Table 1. **Case 1** concerns the fundamental natural frequency of a steel simply supported non-prismatic beam with a I section. The considered beam is composed of two tapered elements, in which the web height is made to increase linearly from 360mm at the supports to 600mm in the mid-span while, the top and bottom flanges remain constant along the beam. In **Case 2**, the free vibration analysis of a non-prismatic cantilever beam is investigated. The web height (the distance between the flange mid-lines) of the cross-section is made to vary linearly from 600 mm at the clamped end to 300 mm at the free end. The beam also exhibits a linear flange tapering. The flange width is ranging along the beam's length from 180 mm to 144 mm, as shown in the following table. The modulus of elasticity and the density of material are assumed $210GPa$ and $7850Kg/m^3$, respectively. **Case 3** gives the first natural frequency of a fixed end prismatic beam under free vibration. In this case, beam is on a variable Winkler type elastic foundation $EI = 1.5\times10^5\,Nm^2$, $\rho A = 1.5\times10^3\,kg/m, K(x) = (4x-3x^2+x^3)\times10^5\,N/m$. **Case 4** presents the values of natural frequency of the first mode for the cantilever beam with constant cross-section. This case deals with beam resting on variable two parameter elastic foundations $EI = 500kNm^2$, $K_1(x) = (6x-x^2)\times10^2\,kN, K(x) = (3x-0.5x^2)\times10^3\,kN/m^2$. All the considered beams and their geometric properties are depicted in Table 1.

Table 1: Effects of number of divisions along the beam's length on the natural frequency $\omega(rad/s)$

<table>
<tr><th rowspan="2">Data Case</th><th colspan="2">Finite Difference Method</th><th rowspan="2">References</th></tr>
<tr><th>Number of Segments (n)</th><th>FDM</th></tr>
<tr><td rowspan="4">Case 1

</td><td>10</td><td>10.145</td><td rowspan="4">10.21
Soltani
(2014)

10.12
Ansys</td></tr>
<tr><td>20</td><td>10.162</td></tr>
<tr><td>30</td><td>10.165</td></tr>
<tr><td>40</td><td>10.166</td></tr>
</table>

	50	10.167	
Case 2 	10	8.539	8.652 Soltani (2014) 8.65 Ansys
	20	8.624	
	30	8.639	
	40	8.645	
	50	8.648	
Case 3 	10	9.782	10.008 Eisenberger (1994)
	20	9.952	
	30	9.983	
	40	9.994	
	50	9.998	
Case 4 	10	38.796	40.382 Eisenberger (1994)
	20	39.158	
	30	39.223	
	40	39.245	
	50	39.25	

The following outcomes can be expressed after noticing the results represented in Table 1:

An outstanding compatibility between the natural frequencies acquired by current study and those computed from the other benchmark solutions is pinnacle. Even by applying 30 segments in the beam's length according to the suggested finite difference method, the natural frequencies can be reckoned bellow the acceptable error rate (1%).

CONCLUSIONS

In the current study, in order to calculate the natural frequencies of non-prismatic members resting on two parameter elastic foundations, central finite difference approximation method is used to solve the fourth-order differential equation of motion with variable coefficients. Regarding the presented numerical example, it can be concluded that by discretizing the considered member into 30-40 divisions the natural frequencies of non-uniform members can be determined through a very good accuracy, within a relative error of 0.1%–0.3%.

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