Input and State Estimation for Linear Discrete-Time Systems without Direct Feedthrough

*Sujuan Luo[2] and Ying Lei[1]

Department of Civil Engineering, Xiamen University, Xiamen 361005, China

Abstract

The classical Kalman filter (KF) is an effective approach of state estimation for linear discrete-time systems, but classical KF is applicable only when external inputs are measured. So far, some studies of Kalman filter with unknown inputs (KF-UI) have been proposed. However, previous KF-UI approaches based solely on acceleration measurements are inherently unstable which leads to poor tracking and fictitious drifts in the identified structural displacements and unknown inputs in the presence of measurement noises. Moreover, it is necessary to have the measurements of acceleration responses at the locations where unknown inputs applied, direct feedthrough of the unknown inputs to the output measurements are required in these approaches. In this paper, it aims to extend the classical KF approach to circumvent the above limitations for Input and state estimation for linear discrete-time systems without direct feedthrough. Based on the scheme of the classical KF, a Kalman filter with unknown excitations (KF-UI) is derived for linear discrete-time systems without direct feedthrough. Then, data fusion of acceleration and displacement or strain measurements is used to prevent the drifts in the identified structural state and unknown inputs.

In addition, dynamic displacement is one of the crucial physical parameters for bridge rating, seismic risk assessment, structural health monitoring of structures. However, it is challenging to measure dynamic displacement because displacement is a relative quantity and requires a fixed reference point Some researchers have investigated dynamic displacement estimation by fusing biased high-sampling rate acceleration and low-sampling rate non-contact displacement measurements. In this

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[1] Professor, Corresponding author, mail: *ylei@xmu.edu.cn*

[2] Graduate Student

paper, the proposed KF-UI is used to identify the dynamic displacement by fusing biased high-sampling rate acceleration and low-sampling rate displacement measurements with consideration of bias in acceleration measurements as “unknown input”. Numerical examples are used to demonstrate the effectiveness of the proposed approach.

1. Introduction

The identification of structural dynamic systems using the measurements of structural vibration data is essential for structural health monitoring and vibration control (Li and Chen 2013; Yuen and Mu 2015). As it is impractical to measure all structural responses, structural identification using only partial measurements of structural responses have received great attentions (Xu *et al*. 2015; Lei *et al.* 2014, 2015). In this regard, the Kalman filter (KF), which was proposed by R.E. Kalman in the early sixties (Kalman 1960), provides a particularly practical and efficient state estimation algorithm with partial measurements of structural responses. However, in the classical KF approach, it is requested that all external inputs are known.

To circumvent the limitation of the classical KF approach, many improved approaches have been proposed for Kalman filter based identification of joint structural state and external inputs, e.g., an iterative identification procedure consisting of the least-squares identification technique and a modification process between each iterative step (Chen and Li 2004); the unbiased minimum-variance input and state estimation with direct feed through (Gillijns and Moor 2007a); a Kalman filter with unknown inputs approach derived by the weighted least-squares estimation method (Pan et al. 2010); a two-stage Kalman estimator in which the classical Kalman filter is first adopted to establish a regression model between the residual innovation and then a recursive least-squares estimator is proposed to identify the input excitation forces (Liu *et al.* 2000; Ma *et al.* 2003; Wu *et al*. 2009); an augmented Kalman filter (AFK) for force identification in structural dynamics, in which the unknown forces are included in the state vector and estimated in conjunction with the states (Lourens *et al*..2012); an average acceleration discrete algorithm with regularization (Ding *et al.* 2013) and implicit Newmark- algorithm with regularization (Liu *et al*. 2014); a weighted adaptive iterative least-squares estimation with incomplete measured excitations (Xu et al. 2015); Kalman estimator with unknown inputs (Lei *et al*. 2012, 2014) and a two-stage and two-step algorithm (Lei *et al.* 2015). However, it has been demonstrated that previous Kalman filter with unknown input (KF-UI) using limited number of acceleration measurements are inherently unstable which leads poor tracking and so-called drifts in the estimated unknown external inputs and structural displacements (Azam *et al.* 2015; Naets et al. 2015). Although regularization approaches (Ding *et al*. 2013; Liu *et al*. 2015) or post-signal processing schemes (Lei *et al.* 2012, 2014, 2015) can be used to treat the drift in the identified results, these treatments prohibits the on-line and real-time identification of coupled structural state and unknown inputs.

Recently, the authors have proposed an improved Kalman filter with unknown inputs based on data fusion of partial acceleration and displacement measurements for real time estimation of joint structural states and the unknown inputs (Liu et al. 2016). However, like other previous Kalman filter with unknown input (KF-UI), it is necessary to have the measurements of acceleration responses at the locations where unknown inputs applied, i.e., direct feedthrough of the unknown inputs to the output measurements are requested. Although Gillijns and Moor investigated unbiased minimum-variance input and state estimation for linear discrete-time systems without direct feedthrough (Gillijns and Moor 2007b), their derivations are quite complex.

In this paper, it aims to extend the classical KF approach and overcome the drawbacks of existing KF-UI approaches for real time estimation of structural states and unknown inputs without direct feedthrough of the unknown inputs to the output measurements. Since accelerations and displacements contains high and low frequencies vibration characteristics, respectively (Smith et al. 2007; Ay and Wang 2014; Kim and Sohn 2014), data fusion of acceleration and displacement or strain measurements is used to prevent the low-frequency drifts in the identified structural state vector and unknown external inputs. Numerical examples of the identification of joint structural state and unknown inputs of and a plane truss are used to demonstrate the effectiveness and versatilities of the proposed algorithm.

What's more, the proposed method is used for real-time dynamic displacement estimation by fusing biased high-sampling rate acceleration and low-sampling rate displacement measurements with the consideration of bias in acceleration measurements as “unknown inputs”. A numerical example is used to demonstrate the effectiveness of the proposed approach.

2. The proposed approach

The equation of motion of a linear discrete-time systems unknown external inputs can be described by

$$\boldsymbol{X}_{k+1} = \boldsymbol{A}_k \boldsymbol{X}_k + \boldsymbol{B}_k \boldsymbol{f}_k^u + \boldsymbol{w}_k \tag{1}$$

where $\boldsymbol{X}_{k+1}$ is the state vector at time $t = k\Delta t$ with Δt being the sampling time step. $\boldsymbol{A}_k$ is the state transformation matrix, $\boldsymbol{B}_k$ is the influence matrix of unknown input vector $\boldsymbol{f}_k^u$, and wk is the model uncertainty which is assumed a noise with zero mean and a covariance matrix $\boldsymbol{Q}_k$.

It is assumed that there is no direct feedthrough of the unknown inputs to the output measurements. Therefore, the discrete observation equation is described by:

$$\boldsymbol{Y}_{k+1} = \boldsymbol{C}_{k+1} \boldsymbol{X}_{k+1} + \boldsymbol{v}_{k+1} \tag{2}$$

where $\boldsymbol{Y}_{k+1}$ is the measured response vector at time $t=(k+1)\Delta t$, $\boldsymbol{C}_{k+1}$ is an known measurement matrices associated with structural state vector, respectively, and $\boldsymbol{v}_{k+1}$ is the measurement noise vector, which is assumed a Gaussian white noise vector with zero mean and a covariance matrix $\boldsymbol{R}_{k+1}$.

Analogous to the classical KF scheme, the proposed KF-UI also contains two procedures. First, $\tilde{\boldsymbol{X}}_{k+1|k}$ is predicted as,

$$\tilde{\boldsymbol{X}}_{k+1|k} = \boldsymbol{A}_k \hat{\boldsymbol{X}}_{k|k} + \boldsymbol{B}_k \hat{\boldsymbol{f}}^u_{k|k} \tag{3}$$

where $\tilde{\boldsymbol{X}}_{k+1|k}$, $\hat{\boldsymbol{X}}_{k|k}$ and $\hat{\boldsymbol{f}}^u_{k|k}$ denote the predicted $\boldsymbol{X}_{k+1}$, estimated $\boldsymbol{X}_k$ and the estimated $\boldsymbol{f}^u$ at time at time $t = k\Delta t$, respectively.

Then, the estimated $\boldsymbol{X}_{k+1}$ in the measurement update (correction) procedure is derived as

$$\hat{\boldsymbol{X}}_{k+1|k+1} = \tilde{\boldsymbol{X}}_{k+1|k} + \boldsymbol{K}_{k+1}(\boldsymbol{Y}_{k+1} - \boldsymbol{C}_{k+1}\tilde{\boldsymbol{X}}_{k+1|k}) \tag{4}$$

where $\hat{\boldsymbol{X}}_{k+1|k+1}$ is the estimated, $\boldsymbol{X}_{k+1}$ given the observations $(\boldsymbol{Y}_1, \boldsymbol{Y}_{2,\ldots,} \boldsymbol{Y}_{k+1})$, $\boldsymbol{K}_{k+1}$ is the Kalman gain matrix which can be derived as

$$\boldsymbol{K}_{k+1} = \tilde{\mathbf{P}}_{k+1|k+1}\boldsymbol{C}^{\mathrm{T}}_{k+1}(\boldsymbol{C}_{k+1}\tilde{\mathbf{P}}_{k+1|k}\boldsymbol{C}^{\mathrm{T}}_{k+1} + \boldsymbol{R}_{k+1})^{-1} \tag{5}$$

in which $\tilde{\mathbf{P}}_{k+1|k}$ is the error covariance of the predicted $\tilde{\boldsymbol{X}}_{k+1|k}$.

Under the condition that the number of measurements (sensors) is no less than that of the unknown inputs, $\hat{\boldsymbol{f}}^u_{k|k}$ can be estimated by minimizing the error vector $\varDelta_{k+1}$ defined by

$$\varDelta_{k+1} = \boldsymbol{y}_{k+1} - \boldsymbol{C}_{k+1}\hat{\boldsymbol{X}}_{k+1|k+1} \tag{6}$$

By inserting the expression of $\hat{\boldsymbol{X}}_{k+1|k+1}$ and $\tilde{\boldsymbol{X}}_{k+1|k}$ in Eq.(4) and Eq.(5) , respectively into the above error vector, $\varDelta_{k+1}$ can be expressed by

$$\varDelta_{k+1} = \left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\left[\boldsymbol{Y}_{k+1} - \boldsymbol{C}_{k+1}\left(\boldsymbol{A}_k\hat{\boldsymbol{X}}_{k|k} + \boldsymbol{B}_k\hat{\boldsymbol{f}}^u_{k|k}\right)\right] \tag{7}$$

Then, $\hat{\boldsymbol{f}}^u_{k|k}$ can be estimated from Eq.(8) based on least-squares estimation as

$$\hat{\boldsymbol{f}}^u_{k|k} = \boldsymbol{M}_{\mathrm{k+1}}\left[\boldsymbol{Y}_{k+1} - \boldsymbol{C}_{k+1}\boldsymbol{A}_k\hat{\boldsymbol{X}}_{k|k}\right] \tag{8}$$

where

$$\boldsymbol{M}_{k+1} = \left[((\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1})(\boldsymbol{C}_{k+1}\boldsymbol{B}_k))^T(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1})(\boldsymbol{C}_{k+1}\boldsymbol{B}_k)\right]^{-1}(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}) \tag{9}$$

Due to $\boldsymbol{M}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{B}_k = \boldsymbol{I}$,

$$\boldsymbol{M}_{k+1} = (\boldsymbol{C}_{k+1}\boldsymbol{B}_k)^T\left[(\boldsymbol{C}_{k+1}\boldsymbol{B}_k)(\boldsymbol{C}_{k+1}\boldsymbol{B}_k)^T\right]^{-1} \tag{10}$$

The error of state estimation defined as $\hat{\boldsymbol{e}}^{\boldsymbol{X}}_{k+1|k+1} = \boldsymbol{X}_{k+1} - \hat{\boldsymbol{X}}_{k+1|k+1}$ can be derived from Eqs.(2-4) as

$$\hat{\boldsymbol{e}}^{\boldsymbol{X}}_{k+1|k+1} = \left(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\right)\tilde{\boldsymbol{e}}^{\boldsymbol{X}}_{k+1|k} - \boldsymbol{K}_{k+1}\boldsymbol{v}_{k+1} \tag{11}$$

where $\tilde{\boldsymbol{e}}^{\boldsymbol{X}}_{k+1|k}$ is defined as $\tilde{\boldsymbol{e}}^{\boldsymbol{X}}_{k+1|k} = \boldsymbol{X}_{k+1} - \tilde{\boldsymbol{X}}_{k+1|k}$. Then,

$$\tilde{\boldsymbol{e}}^{\boldsymbol{X}}_{k+1|k} = \boldsymbol{A}_k\hat{\boldsymbol{e}}^{\boldsymbol{X}}_{k|k} + \boldsymbol{B}_k\hat{\boldsymbol{e}}^{\boldsymbol{f}}_{k|k} + \boldsymbol{w}_k \tag{12}$$

where $\hat{\boldsymbol{e}}^{\boldsymbol{f}}_{k|k}$ is error of estimated $\hat{\boldsymbol{f}}^{u}_{k|k}$ defined as $\hat{\boldsymbol{e}}^{\boldsymbol{f}}_{k|k} = \boldsymbol{f}^{u}_{k} - \hat{\boldsymbol{f}}^{u}_{k|k}$. By inserting $\boldsymbol{Y}_{k+1}$ in Eq.(2) into Eq.(8), $\hat{\boldsymbol{e}}^{\boldsymbol{f}}_{k|k}$ can be derived by

$$\hat{\boldsymbol{e}}^{\boldsymbol{f}}_{k|k} = -\boldsymbol{M}_{k+1}\left[\boldsymbol{C}_{k+1}\boldsymbol{A}_k\hat{\boldsymbol{e}}^{\boldsymbol{X}}_{k|k} + \boldsymbol{C}_{k+1}\boldsymbol{w}_k + \boldsymbol{v}_{k+1}\right] \tag{13}$$

From Eq.(12), the error covariance matrix $\hat{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k+1}$ is estimated as

$$\hat{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k+1} = \left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\tilde{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k}\left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)^{\mathrm{T}} + \boldsymbol{K}_{k+1}\boldsymbol{R}_{k+1}\boldsymbol{K}^{\mathrm{T}}_{k+1} \tag{14}$$

To minimize the error covariance matrix $\hat{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k+1}$, $\boldsymbol{K}_{k+1}$ should be selected as

$$\boldsymbol{K}_{k+1} = \tilde{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k}\boldsymbol{C}^{T}_{k+1}(\boldsymbol{C}_{k+1}\tilde{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k}\boldsymbol{C}^{T}_{k+1} + \boldsymbol{R}_{k+1})^{-1} \tag{15}$$

Then, $\hat{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k+1}$ in Eq.(14) can be simplified as:

$$\hat{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k+1} = \left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\tilde{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k} \tag{16}$$

The error covariance matrix $\tilde{\boldsymbol{P}}^{x}_{k+1|k}$ is expressed as:

$$\tilde{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k} = \left[\boldsymbol{A}_k \;\; \boldsymbol{B}_k\right]\begin{bmatrix}\hat{\boldsymbol{P}}^{\boldsymbol{X}}_{k|k} & \hat{\boldsymbol{P}}^{\boldsymbol{Xf}}_{k|k} \\ \hat{\boldsymbol{P}}^{\boldsymbol{fX}}_{k|k} & \hat{\boldsymbol{P}}^{\boldsymbol{f}}_{k|k}\end{bmatrix}\begin{bmatrix}\boldsymbol{A}^{\mathrm{T}}_{k} \\ \boldsymbol{B}^{T}_{k}\end{bmatrix} + \boldsymbol{Q}_{k+1} \tag{17}$$

The error covariance matrix $\hat{\boldsymbol{P}}^{\boldsymbol{f}}_{k|k}$ can be estimated from Eq.(13) as

$$\hat{\boldsymbol{P}}^{\boldsymbol{f}}_{k|k} = \boldsymbol{M}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{A}_k\hat{\boldsymbol{P}}^{\boldsymbol{X}}_{\boldsymbol{k|k}}\left(\boldsymbol{M}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{A}_k\right)^{\mathrm{T}} + \boldsymbol{M}_{k+1}\boldsymbol{R}_{k+1}\boldsymbol{M}^{\mathrm{T}}_{k+1} + \boldsymbol{M}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{Q}_k\left(\boldsymbol{M}_{k+1}\boldsymbol{C}_{k+1}\right)^{\mathrm{T}} \tag{18}$$

where $\hat{\boldsymbol{P}}^{\boldsymbol{Xf}}_{k|k}$ and $\hat{\boldsymbol{P}}^{\boldsymbol{fX}}_{k|k}$ are the two error covariance matrices defined as

$$\hat{\boldsymbol{P}}^{\boldsymbol{Xf}}_{k|k} = \mathrm{E}\left(\boldsymbol{e}^{\boldsymbol{X}}_{k|k}\boldsymbol{e}^{\boldsymbol{f}\,\mathrm{T}}_{k|k}\right); \quad \hat{\boldsymbol{P}}^{\boldsymbol{fX}}_{k|k} = \mathrm{E}\left(\boldsymbol{e}^{\boldsymbol{f}}_{k|k}\boldsymbol{e}^{\boldsymbol{X}\,\mathrm{T}}_{k|k}\right) \tag{19}$$

and the error covariance matrix $\hat{\boldsymbol{P}}^{\boldsymbol{Xf}}_{k|k}$ can be derived as

$$\begin{aligned}\hat{\boldsymbol{P}}^{\boldsymbol{Xf}}_{k|k} = &-\left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\left(\boldsymbol{A}_k\hat{\boldsymbol{P}}^{\boldsymbol{X}}_{k|k}\boldsymbol{A}^{T}_{k} + \boldsymbol{B}_k\hat{\boldsymbol{P}}^{\boldsymbol{Xf}}_{k|k}\boldsymbol{A}^{T}_{k}\right)\left(\boldsymbol{M}_{k+1}\boldsymbol{C}_{k+1}\right)^{T} \\ &- \boldsymbol{K}_{k+1}\boldsymbol{R}_{k+1}\boldsymbol{M}^{\mathrm{T}}_{k+1} - \left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\boldsymbol{Q}_{k+1}\left(\boldsymbol{M}_{k+1}\boldsymbol{C}_{k+1}\right)^{\mathrm{T}}\end{aligned} \tag{20}$$

and

$$\hat{\boldsymbol{P}}^{\boldsymbol{fX}}_{k|k} = \left(\hat{\boldsymbol{P}}^{\boldsymbol{Xf}}_{k|k}\right)^{\mathrm{T}} \tag{21}$$

3. Numerical validations of the proposed KF-UI

3.1 Identification of a truss-structure and unknown input

To validate the proposed KF-UI for the identification of other type structures with unknown inputs, the identification of a plane truss and unknown input is studied. As shown in Fig.1, the truss consists of 11 uniform members. The length of each horizontal and inclined bar are 2m, $\sqrt{2}$ m, respectively. Other parameters of the truss are: cross section area A= 7.854×10^{-5} m2, Young's module E= 2×10^{11} pa, mass density of truss member $\rho=7.8\times10^{3}kg/m^{3}$ and the mass is concentrated on each node. The truss is subjected to an unknown input in the vertical direction at node 4. In this example, Rayleigh damping $\boldsymbol{C}=\alpha\boldsymbol{M}+\beta\boldsymbol{K}$ is employed with α=0.6993 and β=0.0011.

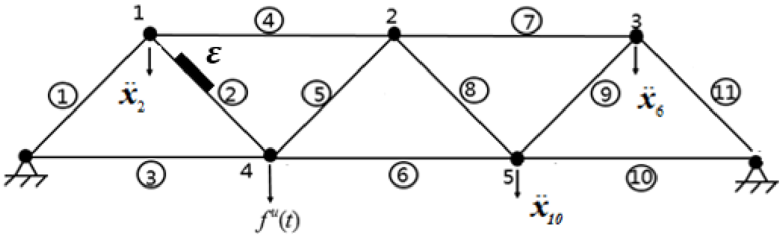
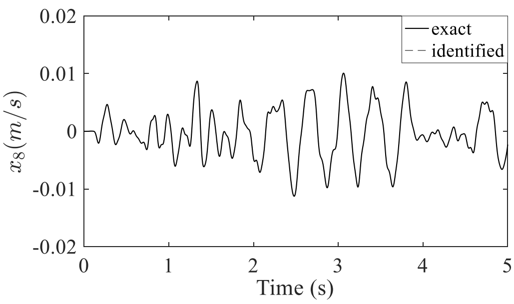
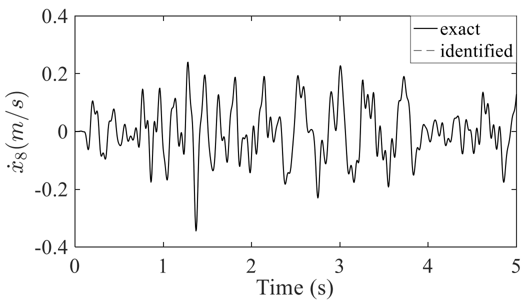


Fig. 1: A plane truss under unknown input

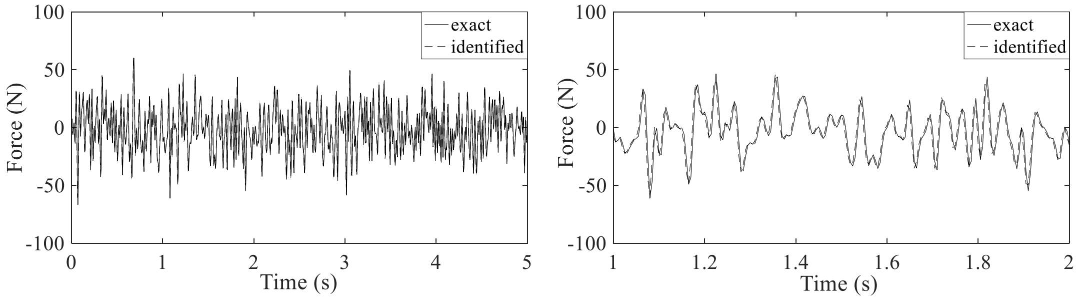
As indicted in Fig.1, acceleration responses in the vertical directions of nodes 1, 3 and 5 are measured. In practice, displacement measurements may be absent but strain measurements are easily available. Displacement measurements can be replaced by strain measurements in the KF-UI based on data fusion. Therefore, partially measured strains are added in combination with the partial acceleration measurements to prevent the above drifts in the identification problem. For this relatively small size structural model, the strain at the second bar in Fig. 4 is measured. Data fusion of this measured strain and the above three accelerations are used in the observation equation. As shown by the comparisons of identified structural state and input with their exact values in Fig. 6.



(a) Identified displacement



(b) Identified velocity



(c) Comparison of identified unknown input

Fig.2 Comparisons of identified results with data fusion

Also, it is noted that identified structural state and unknown input by the proposal algorithm are in good agreements with their corresponding actual values.

3.2 Dynamic displacement by fusing biased high-sampling rate acceleration and low-sampling rate displacement measurements

3.2.1 The proposed method

Some researchers have investigated dynamic displacement estimation by fusing biased high-sampling rate acceleration and low-sampling rate displacement measurements. But, due to the influence of the environment or the sensor itself, the actual measurement of structural responses, there are always measurement bias. In this paper, the acceleration bias can be regarded as an unknown input information, and then use the proposed Kalman Filter (KF-UI) to identify unknown bias and dynamic displacement.

The equation of motion and observation can be written as:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ddot{x}_m + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \eta_a \tag{22}$$

$$z = x_m = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \eta_d \tag{23}$$

where x_m and $\ddot{x}_m$ are the measured displacement and acceleration, the η_d and η_a are the noise of displacement and acceleration.

The state equation of the system can be expressed as:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\ddot{x} + w \tag{24}$$

$$\boldsymbol{z} = \boldsymbol{H}\boldsymbol{x} + v \tag{25}$$

In which $\boldsymbol{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$, $\boldsymbol{w} \sim (0, \boldsymbol{Q})$, $\boldsymbol{Q} = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix}$; $\boldsymbol{v} \sim (0, \boldsymbol{R})$, $\boldsymbol{R} = \mathrm{r}$。

Based on the zero-order holder (ZOH) discretization of the above equation and the consideration of uncertainty in modeling, the state equation of the system in the discrete form can be expressed as:

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_d \boldsymbol{x}_k + \boldsymbol{B}_d \ddot{x}_k + w_k \tag{26}$$

$$\boldsymbol{z}_k = \boldsymbol{H}\boldsymbol{x}_k + v_k \tag{27}$$

If the sampling step of measured responses is T_a， then

$$\boldsymbol{A}_d = \begin{bmatrix} 1 & T_a \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{B}_d = \boldsymbol{G}_d = \begin{bmatrix} T_a^2/4 \\ T_a/2 \end{bmatrix} \tag{28}$$

The acceleration measurement $\ddot{x}_{mk}$ can be expressed as the sum of true acceleration $\ddot{x}_k$ and bias ， namely

$$\ddot{x}_{mk} = \ddot{x}_k + b_k \tag{29}$$

Combine Eq.(27) and (30), it can be obtained that

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_d \boldsymbol{x}_k + \boldsymbol{B}_d \ddot{x}_{mk} - \boldsymbol{B}_d b_k + w_k \tag{30}$$

Let displacement measurement x_{mk} be sampled at T_d and satisfied T_d/T_a=m. Here, m is a sampling rate ratio of displacement measurement to acceleration measurement, and this rate is assumed to be an integer greater than 1 in the proposed multi-rate data fusion scheme. When displacement measurements become available *k=jm*（j=1,2,3···）at the posterior correction step *k=jm*（j=1,2, ···）, the observation equation is obtained as follows:

$$\boldsymbol{z}_k = x_k, \qquad \boldsymbol{H} = [1 \quad 0] \tag{31}$$

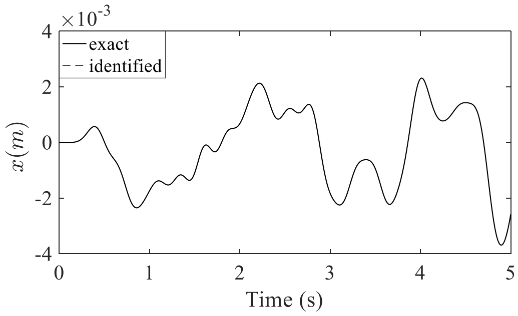
Using the proposed method, the equation(24)can be rewritten as:

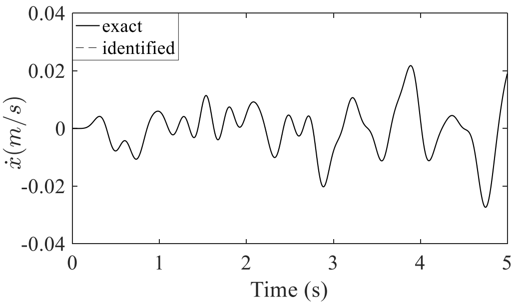
$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_d \boldsymbol{x}_k + \boldsymbol{B}_d f_k - \boldsymbol{B}_d f_k^u + w_k \tag{32}$$

Namely, the known measured acceleration can be equivalent to known force, and the unknown measured acceleration can be equivalent to “unknown input”.

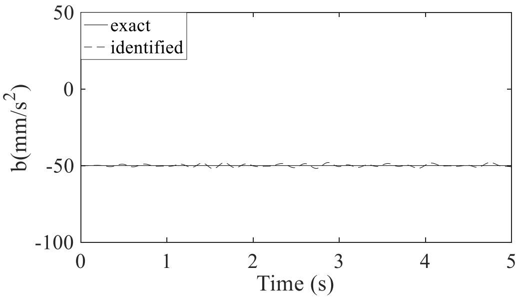
3.2.2 Numerical Simulation

A six-story shear building is used as an example. Parameters of the building are assumed as: floor mass m_i=100kg, floor stiffness k_i=2000 N/m, floor damping c_i=30Ns/m (i=1,2,..,20), respectively. An input of wide-banded white noise is applied to the third floor of the building. Suppose the bias of measured acceleration is -50mm/s ,the time interval T_a=0.001s, T_d=0.01s. As shown by the comparisons of identified dynamic displacement and bias in acceleration measurements with their exact values in Fig. 3.





(a) Identified displacement (b) Identified displacement



(c) Comparison of identified bias

Fig. 3. Comparisons of identified results

Also, it is noted that identified dynamic displacement and unknown bias by the proposal algorithm are in good agreements with their corresponding actual values.

4. Conclusions

Conventional Kalman filter (KF) are applicable only when external inputs are measured. Some improved Kalman filter with unknown inputs (KF-UI) based solely on acceleration measurements are inherently unstable which leads the drifts in the estimated unknown inputs and structural displacements. Moreover, it is necessary to have the measurements of acceleration responses at the locations where unknown inputs applied for the recursive estimation of unknown inputs. In this paper, an algorithm is proposed to circumvent these limitations for the estimation of structural states and unknown inputs of linear discrete-time systems without the direct feedthrough of the unknown inputs to the output measurements. Based on the scheme of the classical KF, a Kalman filter with unknown excitations (KF-UI) and without direct feedthrough is derived. Then, data fusion of acceleration and displacement or strain measurements is used to prevent the drifts in the identified structural state vector and unknown external inputs in real time. The proposed algorithm is not available previous literature and the advantages of the proposed algorithm are obvious since it provides an efficient algorithm of estimation of joint structural states and the unknown inputs.

Moreover, the proposed algorithm can be used for real-time dynamic displacement estimation by fusing biased high-sampling rate acceleration and low-sampling rate displacement measurements with the consideration of bias in acceleration measurements as “unknown inputs”. Numerical examples are used to demonstrate the effectiveness of the proposed approach.

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