







$\phi'_i(t)$  for each  $i$  by refining time frequency curves. As a special reallocation method, synchrosqueezing aims to refine wavelet transform coefficient  $W(t, \omega)$  by assigning its value to different point  $(t', \omega')$  in the time-frequency plane according to the local behavior of  $W(t, \omega)$  around  $(t, \omega)$ .

As the name implies, synchrosqueezing wavelet transform is based on wavelet transform, so it is essential to introduce wavelet transform first. For a given mother wavelet function, the CWT of signal  $x(t)$  is defined by

$$W_x(a, b) = \int_{-\infty}^{\infty} x(t) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{t-b}{a}\right)} dt \quad (2)$$

Where  $a$  and  $b$  is scale factor and dilation factor respectively, and  $\overline{\psi\left(\frac{t-b}{a}\right)}$  represents the complex conjugate of  $\psi\left(\frac{t-b}{a}\right)$ . The mapping between scale factor  $a$  and signal frequency  $\omega$  facilitate displaying wavelet coefficients in time-frequency plane, so several different algorithms can be employed to support the extraction of wavelet ridge and the identification of IFs. Nevertheless, a meaningful research (Daubechies 1996) indicates that wavelet coefficients itself is an oscillating function of time even for the simplest harmonic wave function. If the mother wave function  $\psi$  has fast decay, its Fourier transform  $\hat{\psi}(\xi)$  is approximately equal to zero in the negative frequencies:  $\hat{\psi}(\xi) = 0$ , for  $\xi < 0$ , and is concentrated around  $a = \omega_0/\omega$ . Take  $x(t) = A\cos(\omega t)$  for example, we can rewrite  $W_x(a, b)$  by Plancherel's theorem, as

$$\begin{aligned} W_x(a, b) &= \frac{1}{2\pi} \int \hat{x}(\xi) \sqrt{a} \overline{\hat{\psi}(a\xi)} e^{ib\xi} d\xi = \frac{A}{4\pi} \int [\delta(\xi - \omega) + \delta(\xi + \omega)] \sqrt{a} \overline{\hat{\psi}(a\xi)} e^{ib\xi} d\xi \\ &= \frac{A}{4\pi} \sqrt{a} \overline{\hat{\psi}(a\omega)} e^{ib\omega} \end{aligned} \quad (3)$$

Here,  $\hat{x}(\xi)$  is the Fourier transform of signal  $x(t)$ , while  $\hat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(t) e^{-i\xi t} dt$  represents  $\psi$  in frequency domain. For the purpose of accurately computing the IFs of signals, a research (Daubechies 1996) indicates that although  $W_x(a, b)$  is spread out in  $a$ , its oscillatory behavior in  $b$  points to the original frequency  $\omega$ , no matter what the value of  $a$  would be. Consequently, IF is suggested to be preliminarily estimated by taking derivatives of wavelet coefficients. The formula of computation is shown as follows:

$$\omega_x(a, b) = \begin{cases} \frac{-i\partial_b W_x(a, b)}{W_x(a, b)} & |W_x(a, b)| > 0 \\ \infty & |W_x(a, b)| = 0 \end{cases} \quad (4)$$

Obviously, we can use Eq. (4) to build a map between  $(a, b)$  and  $(\omega_x(a, b),)$  without any difficulty. In the next synchrosqueezing step, the energy from time-scale plane is transferred to the time-frequency plane, according to the map built by Eq. (4).

The frequency variable  $\omega$  and scale factor  $a$  were “binned”, namely discretized, i.e.  $W_x(a, b)$  was computed only at discrete points  $a_i$ , with  $a_i - a_{i-1} = (\Delta a)_i$ , and its synchrosqueezing value was likewise determined merely at the centers  $\omega_l$  of closed intervals  $[\omega_l - \frac{1}{2}\Delta\omega, \omega_l + \frac{1}{2}\Delta\omega]$ , with  $\omega_l - \omega_{l-1} = \Delta\omega$ . By summing these different contributions, synchrosqueezing wavelet transform of  $x(t)$  is obtained.

$$T_x(\omega_l, b) = \sum_{a_i: |\omega_x(a, b) - \omega_l| \leq \Delta\omega/2} W_x(a, b) a_i^{-3/2} (\Delta a)_i \quad (5)$$

If frequency  $\omega$  and scale  $a$  are treated as continuous variables, the analog of the above Eq. is

$$T_x(\omega, b) = \int_{A(b)} W_x(a, b) a^{-3/2} \delta(\omega(a, b) - \omega) da \quad (6)$$

On the surface, synchrosqueezing is similar to TFR methods, however, there are some differences between synchrosqueezing and standard TFR techniques such as STFT, CWT and Wiger-Vill distribution (Thakur 2013). Synchrosqueezing wavelet transform can extract and delineate components by sharpening time-frequency spectrum, and unlike most TFR methods, it allow individual reconstruction of these components. In other words, synchrosqueezing wavelet transform is invertible, so original signal  $x(b)$  can be reconstructed by performing inverse transform to  $T_x(\omega_l, b)$ .

The following argument indicates that the original signal can still be reconstructed after synchrosqueezing wavelet transform is performed. We have

$$\begin{aligned} \int_0^\infty W_x(a, b) a^{-3/2} da &= \frac{1}{2\pi} \int_{-\infty}^\infty \int_0^\infty \hat{x}(\xi) \overline{\hat{\psi}(a\xi)} e^{ib\xi} a^{-1} da d\xi = \frac{1}{2\pi} \int_0^\infty \int_0^\infty \hat{x}(\xi) \overline{\hat{\psi}(a\xi)} e^{ib\xi} a^{-1} da d\xi \\ &= \int_0^\infty \overline{\hat{\psi}(\zeta)} \frac{d\zeta}{\zeta} \cdot \frac{1}{2\pi} \int_0^\infty \hat{x}(\xi) e^{ib\xi} d\xi \end{aligned} \quad (7)$$

By defining a normalizing constant  $C_\psi = \frac{1}{2} \int_0^\infty \overline{\hat{\psi}(\zeta)} \frac{d\zeta}{\zeta}$ , the original signal can be estimated as

$$x(b) = \Re[C_\psi^{-1} (\int_0^\infty W_x(a, b) a^{-3/2} da)] \quad (8)$$

In the piecewise constant approximation corresponding to the binning in  $a$ , Eq. (8) becomes

$$x(b) \approx \Re[C_\psi^{-1} \sum_i W_x(a, b) a_i^{-3/2} (\Delta a)_i] = \Re[C_\psi^{-1} \sum_l T_x(\omega_l, b) (\Delta\omega)] \quad (9)$$

In brief, for a wide range of asymptotic signal according with the synchrosqueezing assumptions, each component of analyzed signal is well concentrated in the time-frequency plane and thus can be estimated successfully, provided a sufficiently fine

division of frequency bins  $\{\omega_l\}$ . More of the synchrosqueezing algorithm (definitions, estimates and proofs) is detailed in the literature (Daubechies 2011).

### 3. Numerical simulations

#### 3.1 IF extraction of Duffing system with free vibration

Classical Duffing Eq. is usually employed to simulate nonlinear motion of mass-spring-damper systems. A Duffing Eq. in this example is given by Feldman (2011).

$$\ddot{x} + 0.05\dot{x} + x + 0.01x^3 = 0 \quad (10)$$

The motion begins with  $x_0 = 10$ ,  $\dot{x} = 0$ , and its response can be simulated using 4<sup>th</sup> Runge-Kutta method with time interval of 0.1 seconds. Fig. 1 shows the results of displacement response.

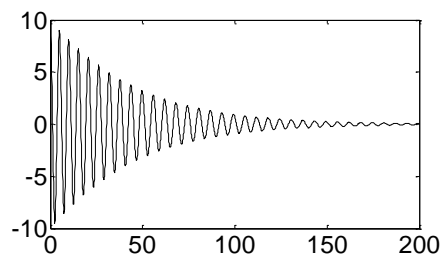


Fig. 1 Displacement response of Duffing system

To verify the influence on synchrosqueezing value  $T_x(\omega, b)$ , Morlet, Gauss and Bump wavelet is chose to conduct CWT, leading to scalogram as Fig. 2. Synchrosqueezing wavelet transform is then performed to extract IF curves according to CWT of response signals. The identified results are shown in Fig. 3.

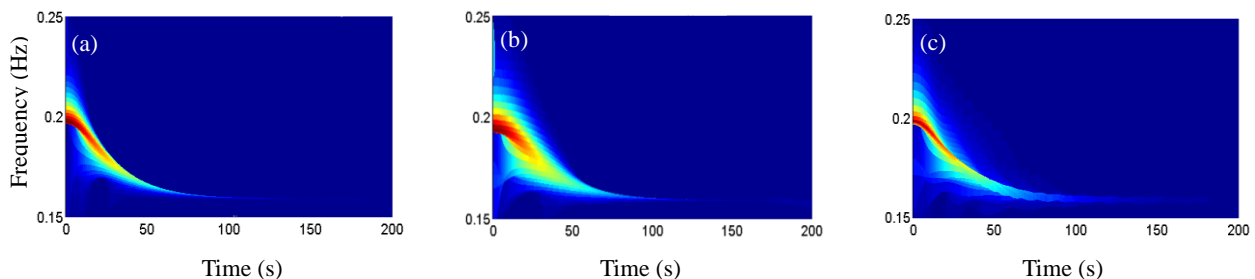


Fig. 2 Wavelet scalogram of displacement response of Duffing system: (a)Morlet wavelet, (b)Gauss wavelet, (c)Bump wavelet



















